College of the Holy Cross, Spring 2014
Math 136, section 1, Midterm Exam 1
Friday, February 21

I. Let $f(x) = x^3 + 1$ on the interval $[a, b] = [1, 3]$.

A. (10) Evaluate the Riemann sum for f on this interval using $n = 4$ and $x_i^* =$ right endpoints.

Solution: With $n = 4$, we have $\Delta x = \frac{3-1}{3} = \frac{1}{2}$. The end points of the intervals are $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$, $x_3 = \frac{5}{2}$, and $x_4 = 3$. The right-hand Riemann sum equals:

$$f(3/2)\Delta x + f(2)\Delta x + f(5/2)\Delta x + f(3)\Delta x,$$

which equals

$$(1/2)[((3/2)^3 + 1) + (2^3 + 1) + ((5/2)^3 + 1) + (3^3 + 1)] = 29$$

B. (10) Now repeat part A, but using the left endpoints.

Solution: Similar to part A, but using the left endpoints:

$$f(1)\Delta x + f(3/2)\Delta x + f(2)\Delta x + f(5/2)\Delta x,$$

which equals

$$(1/2)[(1^3 + 1) + ((3/2)^3 + 1) + (2^3 + 1) + ((5/2)^3 + 1)] = 16$$

C. (5) One of your answers in parts A and B is definitely an overestimate for the value of $\int_1^3 x^3 + 1\, dx$. Which is it?

Solution: Since $f(x) = x^3 + 1$ is increasing on its whole domain, the right-hand sum (part A) gives an overestimate of the integral. (The exact value is

$$\int_1^3 x^3 + 1\, dx = \frac{x^4}{4} + x\Big|_1^3 = \frac{81}{4} + 3 - \frac{1}{4} - 1 = 22.)$$

II. Compute the derivatives of each of the following functions defined by integrals.

A. (5) $f(x) = \displaystyle\int_1^x e^{t^2}\, dt$

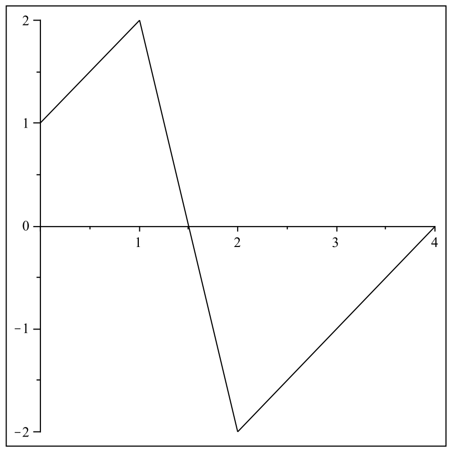
Solution: By the FTC, part 1, $f'(x) = e^{x^2}$

B. (5) $g(x) = \displaystyle\int_{x^2}^4 \frac{\cos(t)}{t^2}\, dt$

Solution: We must reverse the order of the limits, then use the FTC part 1 and the Chain Rule on this one:

$$g(x) = -\int_4^{x^2} \frac{\cos(t)}{t^2}\, dt \Rightarrow g'(x) = \frac{-\cos(x^2)}{x^4} \cdot 2x = \frac{-2\cos(x^2)}{x^3}.$$

III. The following graph (made up of straight line segments) shows $y = f(t)$ for $0 \le t \le 4$.



Given: $f(1) = 2$, $f(2) = -2$, $f(3) = -1$, and $f(4) = 0$. The function F is defined by $F(x) = \int_0^x f(t)\, dt$.

A. (5) Determine the values $F(x)$ for $x = 0, 1, 2, 3, 4$ and enter them in the following table.

Solution: By finding areas of trapezoids and triangles,

x	0	1	2	3	4
$F(x)$	0	3/2	3/2	0	-1/2

B. (5) Does $F(x)$ have any critical points? If so, say where. If not say why not.

Solution: $F'(x) = f(x)$ and this equals 0 when $x = 3/2$ and (in a sense – see below) again when $x = 4$. Of these $x = 3/2$ is definitely a critical point. Whether or not you call $x = 4$ a critical point is sort of a matter of taste (i.e. of the precise way critical points are defined.) Note that $F(x)$ is not defined for $x > 4$, so to define $F'(4)$, one would have to consider just a one-sided limit of the difference quotient. That does exist and equals zero here. So it is OK to say $F'(4) = 0$ in that sense.

C. (5) Over which interval(s) is $F(x)$ concave down?

Solution: This is true on intervals where $F'(x) = f(x)$ is decreasing, so $(1, 2)$

IV.

A. (5) Integrate with a suitable u-substitution: $\displaystyle\int_0^1 (4x^3 + 1)^{3/5} x^2\, dx$.

Solution: Let $u = 4x^3 + 1$. Then $du = 12x^2\ dx$, so $\frac{1}{12} du = x^2\ dx$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 5$. The definite integral goes over to

$$\frac{1}{12} \int_1^5 u^{3/5}\ du = \frac{1}{12} \frac{5}{8}\ u^{8/5} \Big|_1^5 = \frac{5}{96} [5^{8/5} - 1] \doteq .6319$$

B. (10) Integrate with a suitable u-substitution: $\displaystyle\int \frac{x \sin(3x^2)}{\cos(3x^2) + 1}\ dx$.

Solution: Let $u = \cos(3x^2) + 1$. Then $du = -6x \sin(3x^2)\ dx$. So $x \sin(3x^2)\ dx = \frac{-1}{6}\ du$ and the integral goes over to

$$\frac{-1}{6} \int \frac{du}{u} = \frac{-1}{6} \ln|u| + C = \frac{-1}{6} \ln|\cos(3x^2) + 1| + C.$$

C. (15) Integrate by parts: $\displaystyle\int x^2 \sin(5x)\ dx$

Solution: We integrate by parts twice, letting $u =$ power of x each time:

$$\begin{aligned} \int x^2 \sin(5x)\ dx &= \frac{-x^2 \cos(5x)}{5} + \frac{2}{5} \int x \cos(5x)\ dx \\ &= \frac{-x^2 \cos(5x)}{5} + \frac{2}{5} \left(\frac{x \sin(5x)}{5} - \frac{1}{5} \int \sin(5x)\ dx \right) \\ &= \frac{-x^2 \cos(5x)}{5} + \frac{2x \sin(5x)}{25} + \frac{2 \cos(5x)}{125} + C. \end{aligned}$$

D. (10) Integrate with any applicable method we have discussed: $\displaystyle\int_0^1 \frac{x}{\sqrt{x^2 + 1}} dx$

Solution: The $x\ dx$ in the numerator is, up to a constant, the derivative of $u = x^2 + 1$ in the radical:

$$= \frac{1}{2} \int_{u=1}^{u=2} u^{-1/2}\ du = u^{1/2} \Big|_1^2 = \sqrt{2} - 1.$$

E. (10) Integrate with any applicable method we have discussed: $\displaystyle\int e^x \cos(4x)\ dx$

Solution: Use parts twice (letting $dv = e^x\ dx$ each time), then solve for the integral:

$$\begin{aligned} \int e^x \cos(4x)\ dx &= e^x \cos(4x) + 4 \int e^x \sin(4x)\ dx \\ &= e^x \cos(4x) + 4 \left(e^x \sin(4x) - 4 \int e^x \cos(4x)\ dx \right) \\ \text{so } \int e^x \cos(4x)\ dx &= \frac{e^x}{17} (\cos(4x) + 4 \sin(4x)) + C. \end{aligned}$$