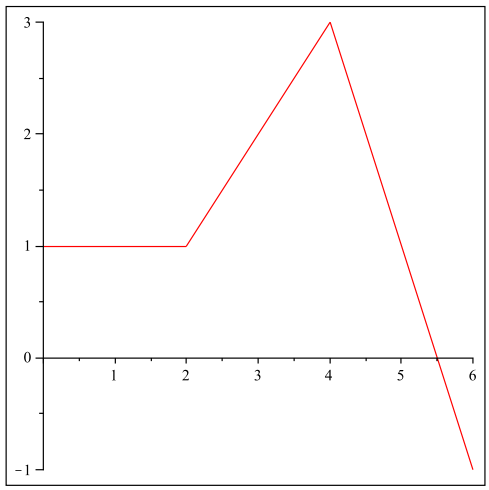
**College of the Holy Cross, Spring Semester, 2014**
**MATH 136, Section 01, Solutions for Final Exam**
**Wednesday, May 14, 11:30 AM**
Professor Little

I.

(A) (10) Let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 2 \\ x-1 & \text{if } 2 < x \leq 4. \\ 11-2x & \text{if } 4 < x \leq 6 \end{cases}$ whose graph is shown here:

**Solution:**




Let $F(x) = \int_0^x f(t)\,dt$, where $f(t)$ is the function from part (B). Complete the following table of values for $F(x)$:

**Solution:** The value $F(x)$ represents the signed area between the graph $y = f(x)$ and the $x$-axis:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $F(x)$ | 0 | 1 | 2 | 3.5 | 6 | 8 | 8 |

(B) (10) Compute the derivative of the function $g(x) = \displaystyle\int_0^{2x} \frac{\cos(t)}{t^2}\,dt$.

**Solution:** By the first part of the Fundamental Theorem of Calculus and the Chain Rule for derivatives:

$$g'(x) = \frac{\cos(2x)}{(2x)^2} \cdot 2 = \frac{\cos(2x)}{2x^2}.$$

II. Compute the following integrals. Some of these may be forms covered by entries in the table of integrals. If you use a table entry, state which one.

(A) (5) $\displaystyle\int \frac{x^4 - 3\pi^3 + \sqrt{x}}{x^{2/3}}\, dx$

**Solution:** Split into separate fractions, simplify and integrate:

$$\begin{aligned}\int \frac{x^4 - 3\pi^3 + \sqrt{x}}{x^{2/3}}\, dx &= \int x^{10/3} - 3\pi^3 x^{-2/3} + x^{-1/6}\, dx \\ &= \frac{3}{13}x^{13/3} - 9\pi^3 x^{1/3} + \frac{6}{5}x^{5/6} + C.\end{aligned}$$

(B) (5) $\displaystyle\int x e^{x^2}\, dx$

**Solution:** Use a $u$-substitution with $u = x^2$, so $du = 2x\, dx$. This gives

$$\int x e^{x^2}\, dx = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

(C) (10) $\displaystyle\int \frac{\csc^2(5x)\, dx}{\cot(5x) + 7}$

**Solution:** This one can also be handled by the $u$-substitution $u = \cot(5x) + 7$, for which $du = -5\csc^2(5x)\, dx$ by the Chain Rule. Then

$$\int \frac{\csc^2(5x)\, dx}{\cot(5x) + 7} = \frac{-1}{5}\int u^{-1}\, du = \frac{-1}{5}\ln|u| + C = \frac{-1}{5}\ln|\cot(5x) + 7| + C.$$

(D) (10) $\displaystyle\int_1^e x^5 \ln(x)\, dx.$

**Solution:** Integrate by parts with $u = \ln(x)$ and $dv = x^5\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{1}{6}x^6$ and by the integration by parts formula,

$$\int x^5 \ln(x)\, dx = \frac{x^6}{6}\ln(x) - \int \frac{1}{6}x^6 \cdot \frac{1}{x}\, dx = \frac{x^6}{6}\ln(x) - \frac{x^6}{36} + C.$$

For the definite integral, we apply the Evaluation Theorem to get

$$\int_1^e x^5 \ln(x)\, dx = \frac{x^6}{6}\ln(x) - \frac{x^6}{36}\bigg|_1^e = \frac{e^6}{6}\ln(e) - \frac{e^6}{36} - \frac{1}{6}\ln(1) + \frac{1}{36} = \frac{5e^6 + 1}{36}.$$

(E) (10) $\displaystyle\int \frac{1}{\sqrt{16 + x^2}}\, dx$

**Solution:** This can be done either with # 25 in the table or by the trigonometric substitution $x = 4\tan\theta$. With the latter approach, $dx = 4\sec^2\theta\,d\theta$, and $\sqrt{16+x^2} = \sqrt{16(1+\tan^2\theta)} = 4\sec\theta$. This simplifies to

$$\int \frac{1}{4\sec\theta}\cdot 4\sec^2\theta\,d\theta = \int \sec\theta\,d\theta.$$

Now we use # 14 in the table (or the memorized form), then convert back to $x$:

$$\begin{aligned}\int \sec\theta\,d\theta &= \ln|\sec\theta+\tan\theta| + C\\ &= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C.\end{aligned}$$

Using properties of logarithms and incorporating $-\ln(4)$ with the constant, this can also be written in the form given in # 25 from the table:

$$\ln|\sqrt{16+x^2}+x| + C.$$

(F) (10) $\displaystyle\int \frac{x}{x^3+x^2+x+1}\,dx$

**Solution:** By partial fractions – we start by factoring $x^3+x^2+x+1 = (x+1)(x^2+1)$. Then we must solve for $A, B, C$ to make:

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Clearing denominators,

$$1 = A(x^2+1) + (Bx+C)(x+1),$$

so equating coefficients, $A+B=0$, $B+C=0$, and $A+C=1$. Solving simultaneously,

$$A = \frac{1}{2}, \qquad B = \frac{-1}{2}, \qquad C = \frac{1}{2}.$$

Then

$$\begin{aligned}\int \frac{1}{(x+1)(x^2+1)}\,dx &= \int \frac{\frac{1}{2}}{x+1} + \frac{\frac{-x}{2}+\frac{1}{2}}{x^2+1}\,dx\\ &= \frac{1}{2}\ln|x+1| + \frac{-1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C.\end{aligned}$$

III.

(A) (5) Set up an integral to compute the length of the curve $y = x^3$ from $x=1$ to $x=3$.

**Solution:** The integral is

$$\int_1^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \sqrt{1+9x^4}\,dx$$

(B) (10) Use a midpoint Riemann sum with $n = 4$ to approximate your integral from part A.

**Solution:** The midpoint Riemann sum approximation is

$$\begin{aligned}\int_1^3 \sqrt{1+9x^4}\,dx &\doteq \left(\sqrt{1+9(1.25)^4}+\sqrt{1+9(1.75)^4}+\sqrt{1+9(2.25)^4}+\sqrt{1+9(2.75)^4}\right)(.5)\\ &\doteq 25.98.\end{aligned}$$

(C) (5)

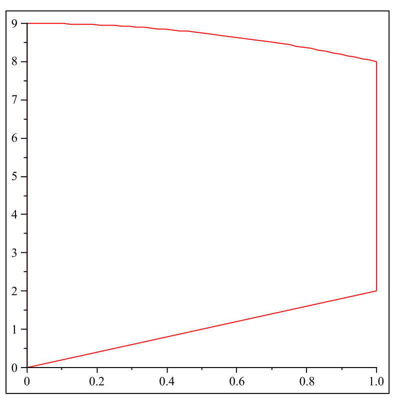
**Solution:** The midpoint approximation is an *underestimate* because $\sqrt{1+9x^4}$ is concave up on $[0, 2]$.

*Note:* From this point on in the exam, if an entry from the table of integrals applies, you may use it for full credit if you state which entry you are using and indicate what $u$ and what constant values $a$, $b$, etc. are involved.

IV. A region $R$ in the plane is bounded by the graphs $y = 9 - x^2$, $y = 2x$, $x = 0$ and $x = 1$.

(A) (20) Compute the area of the region $R$.

**Solution:** Here is a sketch of the region:




The area is given by the integral

$$A = \int_0^1 9 - x^2 - 2x\,dx = 9x - \frac{x^3}{3} - x^2\Big|_0^1 = \frac{23}{3}.$$

(B) (20) Compute the volume of the solid obtained by rotating $R$ about the $x$-axis.

**Solution:** The cross-sections of the solid by planes perpendicular to the $x$-axis are washers with inner radius $r_{in} = 2x$ and outer radius $r_{out} = 9 - x^2$. So the volume is the integral of the area of the cross-section:

$$\begin{aligned} V = \int_0^1 \pi(9-x^2)^2 - \pi(2x)^2\,dx &= \pi \int_0^1 81 - 22x^2 + x^4\,dx \\ &= \pi\left(81x - \frac{22x^3}{3} + \frac{x^5}{5}\bigg|_0^1\right) \\ &= \frac{1108\pi}{15}. \end{aligned}$$

(C) *Extra Credit (10)* Set up the integral(s) to compute the volume of the solid obtained by rotating $R$ about the $y$-axis. *You do not need to compute the value.*

**Solution:** The cross-sections by planes perpendicular to the $y$-axis are all disks, but the function giving the radius of the disk is given by three different formulas depending on whether $0 \le y \le 2$, or $2 \le y \le 8$, or $8 \le y \le 9$.

$$V = \int_0^2 \pi\left(\frac{y}{2}\right)^2\,dy + \int_2^8 \pi(1)^2\,dy + \int_8^9 \pi(\sqrt{9-y})^2\,dy.$$

V. (20) The daily solar radiation $x$ per square meter (in hundreds of calories) in Florida in October has a probability density function $f(x) = c(x-2)(6-x)$ if $2 \le x \le 6$, and zero otherwise. Find value of $c$ and the probability that the daily solar radiation per square meter is greater than 4.

**Solution:** We must have

$$1 = \int_2^6 c(x-2)(6-x)\,dx = c\left(-\frac{x^3}{3} + 4x^2 - 12x\bigg|_2^6\right) = \frac{32c}{3}.$$

Therefore $c = \frac{3}{32}$. Then the probability that $x > 4$ is given by the integral

$$\overline{x} = \int_4^6 \frac{3}{32}(x-2)(6-x)\,dx = \int_4^6 \frac{3x}{4} - \frac{3x^2}{32} - \frac{9}{8}\,dx = \frac{1}{2}.$$

VI. An avian flu epidemic has broken out in Birdsburgh, a large city with total population 10 million. Let $N$ be the number, in millions, of people who have been infected, as a function of time $t$ in weeks. The Birdsburgh Public Health department proposes the model that the rate of change of $N$ is proportional to the product of the number of infected people ($N$) and the number of people not yet infected.

(A) (10) Write the proposed model above as a differential equation, calling the constant of proportionality $k$.

**Solution:** If $N$ people have been infected ($N$ in millions), then the number who have not been infected is $10 - N$ (millions). So the differential equation is

$$\frac{dN}{dt} = kN(10 - N).$$

(B) (5) The function $N(t) = 10/(1 + 9999e^{-t})$ should be a solution of your differential equation from part A. What is the value of $k$?

**Solution:** For this $N$,

$$\frac{dN}{dt} = \frac{-99990e^{-t}}{(1 + 9999e^{-t})^2} \tag{1}$$

and

$$\begin{aligned} N(10 - N) &= \frac{10}{1 + 9999e^{-t}} \cdot \left(10 - \frac{10}{1 + 9999e^{-t}}\right) \\ &= \frac{10 \cdot (-99990e^{-t})}{(1 + 9999e^{-t})^2} \end{aligned} \tag{2}$$

Hence comparing (1) and (2), we must have $k = \frac{1}{10} = .1$. (Note: This could also be done by recognizing that the differential equation can be rearranged to the form of a logistic equation with constant $10k$: $\frac{dN}{dt} = (10k)N\left(1 - \frac{N}{10}\right)$. The general solution of this is

$$N = \frac{10}{1 + ce^{-(10k)t}}$$

So if the given function is a solution we must have $c = 9999$ and $10k = 1$, so $k = .1$.)

C) (10) If the epidemic proceeds according the function given in part B, how many weeks will pass before the number of infected people reaches 1 million?

**Solution:** We must solve $1 = \frac{10}{1+9999e^{-t}}$. So $1 + 9999e^{-t} = 10$, and $t = -\ln(9/9999) = \ln(1111) \doteq 7.0$ weeks. Note that the units of $N$ are millions of people so the left side of the equation is $N = 1$, not $N = 1000000$.

VII. (25) Using the Ratio Test and testing the endpoints, determine the interval of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n x^n}{n^2}$.

**Solution:** We have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{4^{n+1}x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{4^n x^n} \right| &= \lim_{n\to\infty} 4\frac{(n+1)^2}{n^2}|x| \\ &= 4|x|. \end{aligned}$$

We want $4|x| < 1$ or $|x| < \frac{1}{4}$ for absolute convergence, so the radius of convergence is $\frac{1}{4}$. Then at $x = \frac{1}{4}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is the $p$-series with $n = 2$, so it converges. At $x = \frac{-1}{4}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

This is an alternating series with $\frac{1}{n^2}$ decreasing to 0 as $n \to \infty$. So it converges by the Alternating Series Test. The interval of convergence is $\left[-\frac{1}{4}, \frac{1}{4}\right]$.