College of the Holy Cross
MATH 135, section 1
Exam 3 Solutions – Friday, November 22

I. For each of the following functions find the derivative and simplify.

A. (5) $f(x) = e^{2x}\cos(4x)$

Solution: By the product and chain rules,

$$f'(x) = -4e^{2x}\sin(4x) + 2e^{2x}\cos(4x) = 2e^{2x}(\cos(4x) - 2\sin(4x)).$$

B. (5) $g(x) = \dfrac{x^2+2}{\ln(x)}$

Solution: By the quotient rule,

$$g'(x) = \frac{2x\ln(x) - (x^2+2)\frac{1}{x}}{(\ln(x))^2} = \frac{2x^2\ln(x) - x^2 - 2}{x(\ln(x))^2}.$$

C. (5) $h(x) = \tan^{-1}(2x+1)$

Solution: By the derivative rule for inverse tangent and the chain rule,

$$h'(x) = \frac{2}{1+(2x+1)^2}.$$

D. (10) $k(x) = x^{\sin(x)}$

Solution: Since x appears in the base and the exponent, we must use logarithmic differentiation. Writing $y = x^{\sin(x)}$, $\ln(y) = \sin(x)\ln(x)$. So then differentiating implicitly,

$$\frac{y'}{y} = \cos(x)\ln(x) + \frac{\sin(x)}{x}$$

and

$$y' = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right).$$

E. (10) Find y' by implicit differentiation if $x^3y^2 + 2y = 3x$.

Solution:

$$3x^2y^2 + x^3\cdot 2yy' + 2y' = 3,$$

so

$$y' = \frac{3-3x^2y^2}{2x^3y+2}.$$

II. (20) A rocket is launched vertically and is tracked by a ground station 4 miles from the launch pad. What is the vertical speed of the rocket when its height above the ground is 3 miles and its distance to the ground station is increasing at 3300 miles per hour?

Solution: Let y be the height of the rocket above the launch pad as a function of t and z be the distance to the ground station as a function of t. At each time the position of the rocket, the launch pad, and the ground station form a right triangle, so by the Pythagorean theorem we have

$$16 + y^2 = z^2.$$

Differentiating with respect to t:

$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

The vertical speed of the rocket that we want is $\frac{dy}{dt}$. We know that at the time t when $y = 3$, $z = 5$ and $\frac{dz}{dt} = 3300$. Therefore

$$\frac{dy}{dt} = \frac{5 \cdot 3300}{3} = 5500\text{m/hr}.$$

III. All parts of this question refer to the function $f(x) = -2x^3 + 24x + 11$.

A. (10) Find the critical numbers of $f(x)$.

Solution: We have $f'(x) = -6x^2 + 24 = 0$ when $x = \pm 2$. $f'(x)$ exists for all real x, so the only critical numbers are $x = \pm 2$.

B. (10) What does the Second Derivative Test tell you about the behavior of f at each of these critical numbers?

Solution: $f''(x) = -12x$. We see $f''(2) = -24 < 0$, so by the Second Derivative Test, f has a local maximum at $x = 2$. Similarly, $f''(-2) = 24 > 0$. Therefore f has a local minimum at $x = -2$.

IV. All parts of this question refer to the function $f(x) = \dfrac{x}{(2x+1)^2}$, for which the first two derivatives are, in simplified form:

$$f'(x) = \frac{1-2x}{(2x+1)^3}$$
$$f''(x) = \frac{8x-8}{(2x+1)^4}$$

A. (5) What is the domain of $f(x)$?

Solution: Domain of f is all $x \neq -1/2$, or $(-\infty, -1/2) \cup (-1/2, \infty)$.

B. (10) Find all critical numbers and determine where $f(x)$ is increasing and decreasing.

Solution: From the formula for $f'(x)$, we see $f'(x) = 0$ when $x = 1/2$, and $f'(x)$ exists for all $x \neq -1/2, 1/2$. Technical point: $x = -1/2$ is *not* a critical number for f, since critical numbers must be in the domain of the function. So the only critical number is $x = 1/2$. Looking at the sign of $f'(x)$, we see $f'(x) < 0$ for x in $(-\infty, -1/2)$, $f'(x) > 0$ for x in $(-1/2, 1/2)$, and $f'(x) < 0$ for x in $(1/2, \infty)$. Therefore f is decreasing on $(-\infty, -1/2)$ and $(1/2, \infty)$ and increasing on $(-1/2, 1/2)$.

C. (5) What is the absolute minimum value of $f(x)$ on the interval $[0, 3]$?

Solution: The critical number 1/2 is in this interval. We have $f(1/2) = \frac{1}{8}$, $f(0) = 0$ and $f(3) = \frac{3}{49}$. The absolute minimum value is $f(0) = 0$. (Note that by the First Derivative Test, f has a local maximum at $x = 1/2$, so the minimum must be at one of the endpoints.)

D. (5) Determine the concavity of $f(x)$, and find all inflection points.

Solution: $f''(x) > 0$ on $(1, \infty)$ and $f''(x) < 0$ on $(-\infty, -1/2)$ and $(-1/2, 1)$. The concavity changes at $x = 1$, so that is the only inflection point.