College of the Holy Cross, Fall 2013
Math 135, Section 1, Midterm 1 Solutions
Friday, September 20

I. Match the plots below with the following formulas. Note that there is an extra plot.

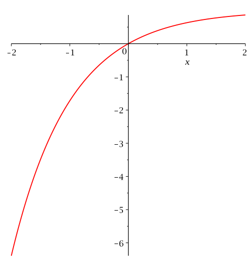
(5) A) $y = 3 - (x-1)^2$ is Plot: $\underline{3}$ (the $x - 1$ shifts the parabola to the right, not the left)

(5) B) $y = \sin(x/2)$ is Plot: $\underline{2}$ (the $x/2$ makes the period equal to $4\pi \doteq 12.6$, so the usual sine graph is stretched horizontally)

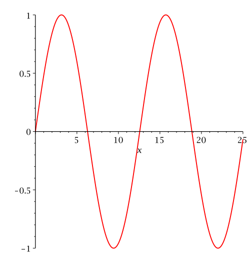
(5) C) $y = 1 - e^{-x}$ is Plot: $\underline{1}$ (think: $y = e^x$ reflected across the x- and y-axes, then shifted up)

(5) D) $y = \sin(2x)$ is Plot: $\underline{5}$ (the $2x$ makes the period equal to $\pi \doteq 3.14$, so the usual sine graph is compressed horizontally).

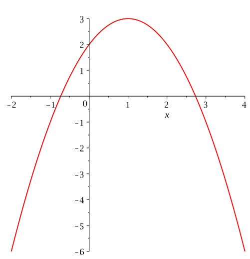
Plot 1:



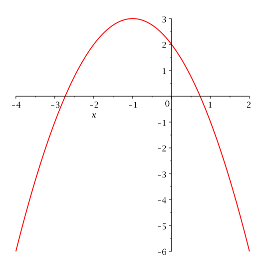
Plot 2:



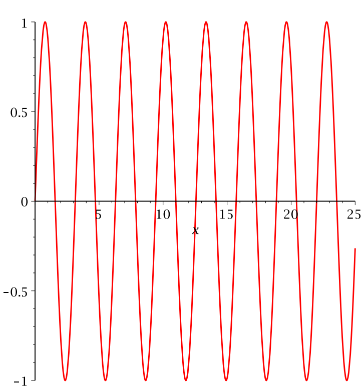
Plot 3:



Plot 4:



Plot 5:



II. The manager of a furniture factory has collected the following data for the cost of manufacturing chairs.

# Chairs (per day) x	Cost (in dollars) y
100	2400
150	3100
250	4500
300	5200

(10) A) Given that y is a linear function of x, determine a formula for it.

The slope is $m = \frac{3100-2400}{150-100} = 14$ so by the point slope form, we get $y - 2400 = 14(x - 100)$, or $y = 14x + 1000$.

Cost function: $y - 2400 = 14(x - 100)$ or $y = 14x + 1000$

(5) B) What does the slope represent in real-world terms?

The slope represents the cost of manufacturing one additional chair per day, or $C(x+1) - C(x)$.

(5) C) What does the y-intercept represent in terms of cost?

The y-intercept of 1000 represents the cost per day if no chairs are actually manufactured ($x = 0$). These are often called *fixed costs* – things like the maintenance costs of the factory, taxes, labor costs, etc.

(5) D) Using your model, determine how much it will cost to produce 350 chairs per day.

Cost: (14)(350) + 1000 = \$ 5900

III. Given $f(x) = 4 - x^2$ and $g(x) = \sqrt{3x-2}$, answer the following questions.

(10) A) Find the domain of $f(x)$ and the domain of $g(x)$.

The domains here are the sets of all real x that can be substituted into the formulas to yield a well-defined result. For f there are no restrictions. For g, we must have $3x - 2 \geq 0$, so $x \geq \frac{2}{3}$.

Domain of f: all real x, or $(-\infty, +\infty)$

Domain of g: all real $x \geq \frac{2}{3}$, or $\left[\frac{2}{3}, +\infty\right)$

(5) B) What is the domain of the function $g(x)/f(x)$?

Now we must be able to substitute an x that makes sense for g, and that also avoids making $f(x) = 0$.

Domain of $g(x)/f(x)$: $\left[\frac{2}{3}, 2\right) \cup (2, +\infty)$, or something equivalent

(5) C) Find the function $g \circ f$. $(g \circ f)(x) = g(f(x)) = \sqrt{3(4 - x^2) - 2} = \sqrt{10 - 3x^2}$

IV. Answer the following questions.

(5) A) Find all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $|\tan x| > 1$.

This is true if $\tan(x) > 1$ or $\tan(x) < -1$. The first occurs for x between $\frac{\pi}{4}$ and $\frac{\pi}{2}$; the second occurs for x between $-\frac{\pi}{2}$ and $-\frac{\pi}{4}$.

Values of x: $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Note: Equivalent answers like: all x with $-\frac{\pi}{4} < x < -\frac{\pi}{4}$ or $\frac{\pi}{4} < x < \frac{\pi}{2}$ are also OK.

(5) B) If $\sin\theta = \dfrac{2}{3}$ and $\dfrac{\pi}{2} < \theta < \pi$, give the exact value of $\cos\theta$.

We can use the basic trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for this: $(\frac{2}{3})^2 + \cos^2(\theta) = 1$ so $\cos^2(\theta) = \frac{5}{9}$ and $\cos(\theta) = \pm\frac{\sqrt{5}}{3}$. Since θ is between $\frac{\pi}{2}$ and π, the cosine must be negative, so the correct answer is:

$\cos\theta$: $-\frac{\sqrt{5}}{3}$

(5) C) Express as a single logarithm: $\dfrac{1}{2}\ln 3 - 3\ln 2 + \ln 6$.

Use the properties of logarithms: $\ln(A) + \ln(B) = \ln(AB)$, $\ln(A) - \ln(B) = \ln(A/B)$, and $p\ln(A) = \ln(A^p)$. Then

$$\frac{1}{2}\ln 3 - 3\ln 2 + \ln 6 = \ln\left(\frac{3^{1/2} \cdot 6}{2^3}\right).$$

Single logarithm: $\ln\left(\frac{3^{1/2} \cdot 6}{2^3}\right) = \ln\left(\frac{3\sqrt{3}}{4}\right)$

V. Consider the function $f(x) = \frac{1}{2}e^{x+1} + 1$.

(15) A) Given that f is one-to-one, find a formula for the inverse function of f.

Set up $y = \frac{1}{2}e^{x+1} + 1$ and solve for x:

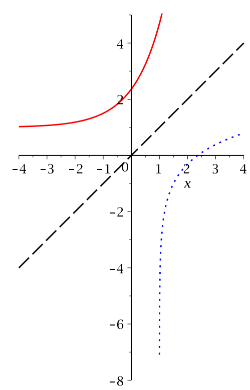
$$\begin{aligned} 2(y-1) &= e^{x+1}, \text{so after taking natural log of both sides} \\ \ln(2(y-1)) &= x+1 \\ x &= \ln(2(y-1)) - 1. \end{aligned}$$

We can swap the variables to write the inverse function as a function of x:

$$f^{-1}(x) = \boxed{\ln(2(x-1)) - 1}$$

(10) B) In the space below, plot the graphs of the functions f and f^{-1} on the same set of axes. Label one point on each graph with its coordinates.

Here are the graphs:



Note that $f(x) > 1$ for all x. This means that $y = f(x)$ should be approaching the horizontal line $y = 1$ as $x \to -\infty$. Because of this, the graph $y = f^{-1}(x)$ has a vertical asymptote at $x = 1$. It is obtained by reflecting $y = f(x)$ across the line $y = x$. The top (red) curve is $y = f(x)$; it contains the point $\left(0, \frac{e}{2} + 1\right) \doteq (0, 2.36)$. The bottom (blue) curve is $y = f^{-1}(x)$, obtained by reflecting $y = f(x)$ across the line $y = x$; it contains the point $\left(\frac{e}{2} + 1, 0\right) \doteq (2.36, 0)$.