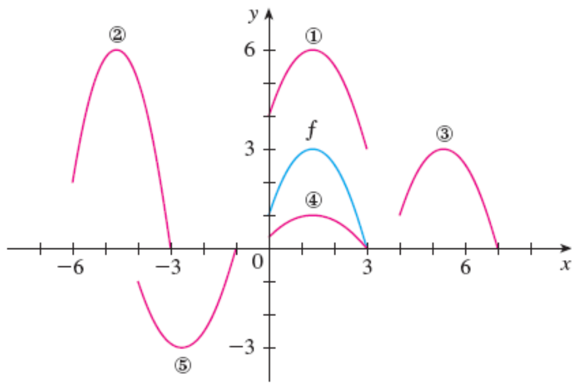
College of the Holy Cross
MATH 135, section 1
Solutions for Final Examination – Thursday, December 12

I. The graph $y = f(x)$ is given in blue. Match each equation with one of the numbered pink graphs.



(5) A) $y = f(x - 4)$ is plot number: 3 (shift 4 units right)

(5) B) $y = f(x) + 3$ is plot number: 1 (shift 3 units up)

(5) C) $y = \frac{1}{3}f(x)$ is plot number: 4 (compress vertically by a factor of $\frac{1}{3}$)

(5) D) $y = -f(x + 4)$ is plot number: 5 (shift 4 units left and reflect across the x-axis)

(5) E) $y = 2f(x + 6)$ is plot number: 2 (shift 6 units lift and stretch vertically by a factor of 2)

II. A cup of hot chocolate is set out on a counter at $t = 0$. The temperature of the chocolate t minutes later is $C(t) = 70 + 80e^{-t/3}$ (in degrees F).

A) (5) What is the temperature of the chocolate at $t = 0$?

Answer: $C(0) = 70 + 80e^{-0/3} = 150$ degrees F.

B) (10) What is the rate of change of the temperature at $t = 10$ minutes?

Solution: The (instantaneous) rate of change at $t = 10$ is $C'(10)$. Since $C'(t) = \frac{-80}{3}e^{-t/3}$ by the chain rule, $C'(10) = \frac{-80}{3}e^{-10/3} \doteq -.95$ degrees F per minute.

Comment: Since the question says "at $t = 10$" you should think: "instantaneous rate of change." Quite a few people in the class computed an average rate of change from $t = 0$ to $t = 10$, which is not the same!

C) (10) How long does it take for the temperature to reach $100°F$?

Solution: The time is the solution of $100 = 70 + 80e^{-t/3}$, or $t = -3\ln(30/80) \doteq 2.9$ minutes.

III. Compute the following limits. Any legal method is OK.

(A) (10) $\lim_{x\to 3} \frac{x^2+x-12}{x^2-5x+6}$.

Solution: Since $x^2 + x - 12 = (x-3)(x+4)$ and $x^2 - 5x + 6 = (x-3)(x-2)$, for $x \neq 3$, the function is

$$\frac{x^2+x-12}{x^2-5x+6} = \frac{x+4}{x-2}.$$

Hence the limit equals

$$\lim_{x\to 3} \frac{x+4}{x-2} = 7$$

by the limit quotient rule. (Note: this could also be done with L'Hopital's Rule.)

(B) (10) $\lim_{x\to 1^-} \frac{|x-1|}{x^2-1}$.

Solution: The denominator is $x^2 - 1 = (x-1)(x+1)$. The numerator is $x - 1$ if $x > 1$ and $-(x-1)$ if $x < 1$. Hence the function equals

$$\begin{cases} \frac{-1}{x+1} & \text{if } x < 1 \\ \frac{1}{x+1} & \text{if } x > 1. \end{cases}$$

This shows that the one-sided limit exists and equals

$$\lim_{x\to 1^-} \frac{-1}{x+1} = \frac{-1}{2}.$$

(The overall limit does not exist since the limit from the other side exists but equals a different value, namely $\frac{+1}{2}$.)

(C) (10) $\lim_{x\to 0^+} x^2 \ln(x)$

Solution: This is indeterminate of the form $0 \cdot \infty$. So we want to rearrange, and then apply L'Hopital's Rule like this:

$$\begin{aligned} \lim_{x\to 0^+} x^2 \ln(x) &= \lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}}, \text{ which is an } \infty/\infty \text{ indeterminate form} \\ &= \lim_{x\to 0^+} \frac{x^{-1}}{-2x^{-3}} \\ &= \lim_{x\to 0^+} \frac{x^2}{-2} \\ &= 0. \end{aligned}$$

(D) (10) $\lim_{x\to 0} \dfrac{\tan(x)}{x^{1/2}}$

Solution: This is indeterminate of the form 0/0. We can apply L'Hopital directly like this:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(x)}{x^{1/2}} &= \lim_{x\to 0} \frac{\sec^2(x)}{\frac{1}{2x^{1/2}}} \\
&= \lim_{x\to 0} 2x^{1/2}\sec^2(x) \\
&= 0 \cdot 1 = 0.
\end{aligned}$$

Comment: I gave some partial credit for experimentation with a table of values on C and D. But you should be aware that that is not a complete justification for saying the limit is zero in C and D. It's suggestive, but it's not a complete reason.

IV.

A) (10) *Using the limit definition*, and showing all necessary steps to justify your answer, compute $f'(x)$ for $f(x) = 5x^2 - x + 3$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{5(x+h)^2 - (x+h) + 3 - 5x^2 + x - 3}{h} \\
&= \lim_{h\to 0} \frac{10xh + 5h^2 - h}{h} \\
&= \lim_{h\to 0} 10x - 1 + 5h \\
&= 10x - 1.
\end{aligned}$$

IV. (continued) Using appropriate derivative rules, compute the derivatives of the following functions. You do not need to simplify your answers.

B) (5) $g(x) = 4x^3 + \sqrt{x} + \dfrac{2}{\sqrt[4]{x}} + e^2$.

Solution: We can rewrite $g(x)$ as

$$g(x) = 4x^3 + x^{1/2} + 2x^{-1/4} + e^2.$$

So by the power and sum rules for derivatives

$$g'(x) = 12x^2 + \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-5/4} + 0.$$

C) (10) $h(x) = \dfrac{\sin(x) + x}{\sec(x)}$.

Solution: By the quotient rule,

$$h'(x) = \frac{\sec(x)(\cos(x) + 1) - (\sin(x) + x)\sec(x)\tan(x)}{\sec^2(x)}.$$

D) (10) $i(x) = \ln(x^3 + 3)$.

Solution: By the chain rule,

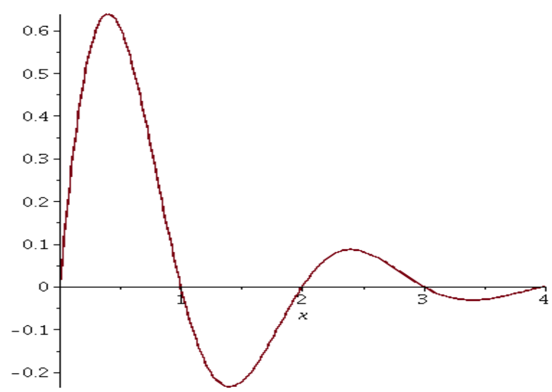
$$i'(x) = \frac{3x^2}{x^3 + 3}.$$

E) (10) $j(x) = \tan^{-1}(12x + 2) + x^x$

Solution: By the derivative rules for the inverse tangent and the chain rule, plus logarithmic differentiation for the x^x,

$$j'(x) = \frac{12}{1 + (12x + 2)^2} + x^x(1 + \ln(x)).$$

V. The following graph shows the *derivative* $f'(x)$ for some function $f(x)$ defined on $0 \le x \le 4$. Note: This *is not* $y = f(x)$, it is $y = f'(x)$.



Using the graph, *estimate*

A) (5) The interval(s) on which $f(x)$ is increasing.

Solution: $f(x)$ is increasing on intervals where $f'(x) > 0$. Here that is true for x in $(0, 1)$ and $(2, 3)$.

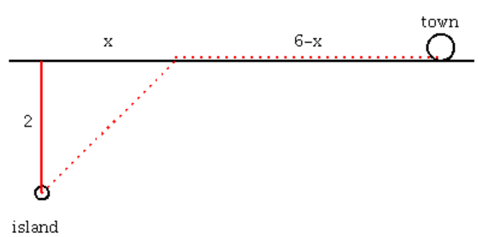
B) (10) The critical numbers of $f(x)$ *in the open interval* $(0, 4)$. Say what the behavior of $f(x)$ is at each critical number (local max, local min, neither).

Solution: The critical numbers in this interval are the places where $f'(x) = 0$, so $x = 1, 2, 3$. By the First Derivative Test, f has local maxima at $x = 1$ and $x = 3$ (f' goes from positive to negative), while f has a local minimum at $x = 2$ (f' goes from negative to positive).

C) (10) The interval(s) on which $y = f(x)$ is concave down.

Solution: f is concave down on intervals where $f''(x) < 0$, or equivalently where $f'(x)$ is decreasing. That is true here for x in $(.4, 1.3)$ and again for x in $(2, 4, 3.3)$ (approximately).

VI. A town wants to build a pipeline from a water station on a small island 2 miles from the shore of its water reservoir to the town. One possible route is shown dotted in red. The town is 6 miles along the shore from the point nearest the island. It costs \$3 million per mile to lay pipe under the water and \$2 million per mile to lay pipe along the shoreline.



A) (5) Give the cost $C(x)$ of constructing the pipeline as a function of x.

Solution: By the Pythagorean theorem and the given information about cost per mile, we have

$$C(x) = 3\sqrt{4 + x^2} + 2(6 - x)$$

1. B) (10) Where along the shoreline should the pipeline hit land to minimize the costs of construction?

Solution: To find the minimum of $C(x)$, we can restrict to x in the closed interval $[0, 6]$, since it clearly does no good to take $x < 0$ or $x > 6$. The function $C(x)$ has a critical number for $x > 0$ at the positive solution of $C'(x) = 0$:

$$\begin{aligned} 0 &= \frac{3x}{\sqrt{4 + x^2}} - 2, \text{or} \\ 3x &= 2\sqrt{4 + x^2} \\ 9x^2 &= 16 + 4x^2 \\ 5x^2 &= 16 \\ x &= \frac{4}{\sqrt{5}} \doteq 1.79. \end{aligned}$$

We have $C(0) = 18$, $C(6) = 3\sqrt{40} \doteq 19.0$, and $C\left(\frac{4}{\sqrt{5}}\right) \doteq 16.47$. So the minimum cost is attained at $x = \frac{4}{\sqrt{5}} \doteq 1.79$ miles.

VII. (15) A block of dry ice (solid CO_2) is evaporating and losing volume at the rate of 10 cm^3/min. It has the shape of a cube at all times. How fast are the edges of cube shrinking when the block has volume 216 cm^3?

Solution: Call the side of the cube x. Then $V = x^3$. Taking time derivatives, we have $V' = 3x^2x'$. From the given information, when $V = 216$, $x = 6$ and $V' = -10$. Therefore the rate of change of the side of the cube is

$$x' = \frac{-10}{3 \cdot 6^2} = \frac{-5}{54} \doteq -.093$$

(units cm/min). The side of the cube is decreasing at about .09 cm/min.

VIII. (10) True or false: The graph obtained by stretching $y = e^{-x}$ vertically by a factor of 2 can *also* be obtained from $y = e^{-x}$ by a horizontal shift. Explain your answer.

Solution: This is TRUE, because

$$2e^{-x} = e^{\ln(2)}e^{-x} = e^{-(x-\ln(2))}.$$

So exactly the same graph is obtained if we stretch $y = e^{-x}$ vertically by a factor of 2, or shift $y = e^{-x}$ to the right by $\ln(2)$ units. This seems counterintuitive, but it is a general property of exponential functions that this sort of thing is true(!)