The Planar, Circular, Restricted Four-Body Problem

Gareth E. Roberts   Julianne Kulevich   Christopher J. Smith

Department of Mathematics and Computer Science
College of the Holy Cross

NSF DMS-0708741

HC Faculty Research Seminar
April 17, 2008

Figure: Julie Kulevich – Second place finisher in the HC Iron Chef competition (Worcester T & G, 4/16/08)

The Planar n-Body Problem

$$\begin{aligned}
m_i &= \text{mass of the } i\text{-th body} \\
\mathbf{q}_i &= \text{position of the } i\text{-th body in } \mathbb{R}^2 \\
\mathbf{p}_i &= m_i\, \dot{\mathbf{q}}_i \quad \text{(momentum)} \\
r_{ij} &= ||\mathbf{q}_i - \mathbf{q}_j|| \quad \text{(mutual distance)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{q} &= (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in \mathbb{R}^{2n} \\
\mathbf{p} &= (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{2n} \\
M &= \text{diag}\,\{m_1, m_1, m_2, m_2, \ldots, m_n, m_n\}
\end{aligned}$$

Newtonian potential function:

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

Equations of motion:

$$
\begin{aligned}
m_i \ddot{\mathbf{q}}_i &= \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i \in \{1, 2, \ldots n\} \\
&= \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3}
\end{aligned}
$$

Hamiltonian system:

$$
\begin{aligned}
\dot{\mathbf{q}} &= M^{-1} \mathbf{p} = \frac{\partial H}{\partial \mathbf{p}} \\
\dot{\mathbf{p}} &= \nabla U(\mathbf{q}) = -\frac{\partial H}{\partial \mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
H(\mathbf{q}, \mathbf{p}) &= K(\mathbf{p}) - U(\mathbf{q}) \\
K(\mathbf{p}) &= \sum_{i=1}^{n} \frac{||\mathbf{p}_i||^2}{2m_i} \qquad \text{Kinetic Energy}
\end{aligned}
$$

Equilibria?

- For $(\mathbf{q}, 0)$ to be an equilibrium point, $\nabla U(\mathbf{q}) = 0$.

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

- But, U is a homogeneous potential of degree -1

$$\nabla U(\mathbf{q}) \cdot \mathbf{q} \ = \ -U(\mathbf{q}) \ < \ 0$$

- Therefore, there are no equilibrium points.
- Physically, this is to be expected.

Periodic Solutions

Complexify and guess a solution of the form

$$\mathbf{q}(t) = \phi(t)\,\mathbf{x} \qquad \text{with } \phi(t) : \mathbb{R} \mapsto \mathbb{C}, \mathbf{x} \in \mathbb{C}^n$$

This leads to

$$\ddot{\phi} = -\frac{\mu|\phi|}{\phi^3} \qquad \text{2d Kepler problem}$$

and

$$\nabla U(\mathbf{x}) + \mu\, M\mathbf{x} = 0. \tag{1}$$

A vector **x** satisfying equation (1) is called a **planar central configuration** (c.c.). Attaching a particular solution of the Kepler problem to each body in a planar c.c. yields a solution to the full *n*-body problem.

- Rigid rotations (same shape and size)
- Elliptical periodic orbits (same shape, oscillating size)

Definition

A **relative equilibrium** for the *n*-body problem is a solution of the form

$$\mathbf{q}(t) = R(\omega t)\mathbf{x}$$

(a rigid rotation) where

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

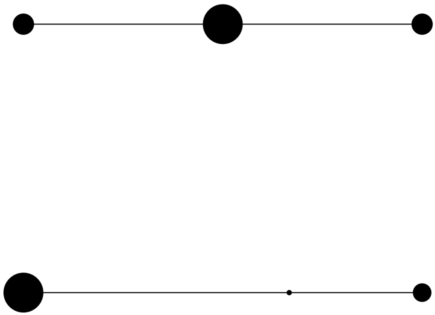
and

$$R(t)\mathbf{q} = (R(t)\mathbf{q}_1, R(t)\mathbf{q}_2, \ldots, R(t)\mathbf{q}_n).$$

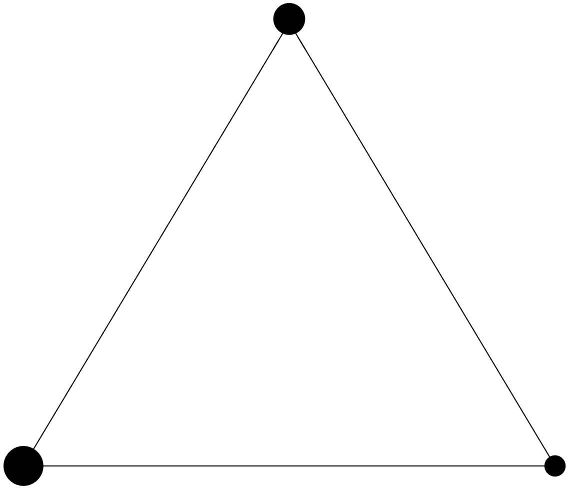
In order to have a relative equilibrium:

- $\mathbf{x}$ must be a planar cc, that is, $\nabla U(\mathbf{x}) + \mu M\mathbf{x} = 0$
- $\omega^2 = \mu = \frac{U(\mathbf{x})}{\sum m_i ||\mathbf{x}_i||^2}$ (rotation speed determined by $\mathbf{x}$)

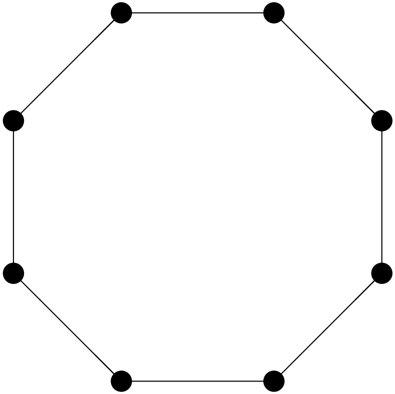
3-Body Collinear Configuration (Euler 1767)



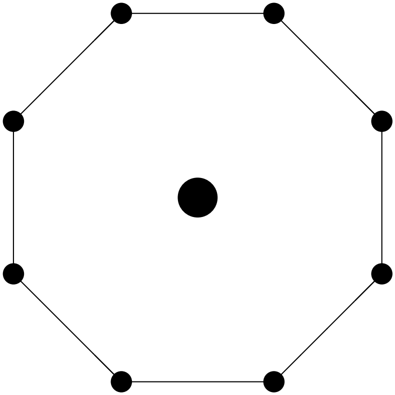
Equilateral Triangle (Lagrange 1772)



Regular n-gon (equal mass required for $n \geq 4$)



1 + n-gon (arbitrary central mass)



Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings (winner of the Adams Prize)

Degeneracies

$$\sum_{j \neq i} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^3} + \omega^2 \, m_i \mathbf{x}_i = 0, \quad i = \{1, 2, \ldots, n\} \tag{2}$$

$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is a relative equilibrium implies that

$$c\mathbf{x} = (c\mathbf{x}_1, c\mathbf{x}_2, \ldots, c\mathbf{x}_n) \quad \text{and}$$

$$R\mathbf{x} = (R\mathbf{x}_1, R\mathbf{x}_2, \ldots, R\mathbf{x}_n)$$

are relative equilibria where c is a constant and $R \in SO(2)$.

The **moment of inertia** $I(\mathbf{x})$ is defined as

$$I(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{n} m_i \|\mathbf{x}_i\|^2$$

Equation (2) for a relative equilibrium can be viewed as a Lagrange multiplier problem: $(I(\mathbf{x}) = k)$

$$\nabla U(\mathbf{x}) + \omega^2 \, \nabla I(\mathbf{x}) = 0.$$

A Topological Viewpoint

Let S be the ellipsoid defined by $2I = 1$ (fixes scaling). Define an equivalence relation via $\mathbf{x} \sim R\mathbf{x},\ R \in SO(2)$ (identify rotationally equivalent relative equilibria).

Critical points of $U([\mathbf{x}])$ on $S/\sim$ are relative equilibria.

Smale/Wintner/Chazy Question: Is the number of relative equilibria equivalence classes finite? (Smale's 6th problem for the 21st century)

A Topological Viewpoint

Let S be the ellipsoid defined by $2I = 1$ (fixes scaling). Define an equivalence relation via $\mathbf{x} \sim R\mathbf{x}, \ R \in SO(2)$ (identify rotationally equivalent relative equilibria).

Critical points of $U([\mathbf{x}])$ on $S/\sim$ are relative equilibria.

Smale/Wintner/Chazy Question: Is the number of relative equilibria equivalence classes finite? (Smale's 6th problem for the 21st century)

- $n = 3$ Euler, Lagrange
- $\frac{n!}{2}$ Collinear CC's Moulton
- 4 Equal masses Albouy (1995)
- $n = 4$ Hampton and Moeckel (2006) (BKK Theory)

A Topological Viewpoint

Let S be the ellipsoid defined by $2I = 1$ (fixes scaling). Define an equivalence relation via $\mathbf{x} \sim R\mathbf{x}, \;\; R \in SO(2)$ (identify rotationally equivalent relative equilibria).

Critical points of $U([\mathbf{x}])$ on $S/\sim$ are relative equilibria.

Smale/Wintner/Chazy Question: Is the number of relative equilibria equivalence classes finite? (Smale's 6th problem for the 21st century)

- $n = 3$ Euler, Lagrange
- $\frac{n!}{2}$ Collinear CC's Moulton
- 4 Equal masses Albouy (1995)
- $n = 4$ Hampton and Moeckel (2006) (BKK Theory)
- $n \geq 5$ Open problem!

Another Finiteness Question

Saari's Conjecture (1970) *Every solution of the Newtonian n-body problem that has a constant moment of inertia (constant size) is a relative equilibrium (rigid rotation).*

Another Finiteness Question

Saari's Conjecture (1970) *Every solution of the Newtonian n-body problem that has a constant moment of inertia (constant size) is a relative equilibrium (rigid rotation).*

Results on Saari's Conjecture

- Newtonian 3-body problem, equal mass case: Saari's conjecture is true (McCord 2004)
- Newtonian 3-body problem, any choice of masses, any dimension: Saari's conjecture is true (Moeckel 2005)
- Mutual distance potentials, collinear case: Generalized Saari's conjecture is true (Diacu, Pérez-Chavela, Santoprete 2004)
- 5-body problem for certain potentials, and a negative mass: Generalized Saari's conjecture is false (GR 2006)
- Inverse Square potential: Generalized Saari's conjecture is decidedly false

Mutual Distances Make Great Coordinates

Recall:

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

Alternative formula for I in terms of mutual distances: (center of mass at origin)

$$I(\mathbf{q}) = \frac{1}{2\tilde{M}} \sum_{i<j}^{n} m_i m_j r_{ij}^2$$

Fact: Constant inertia $\Rightarrow$ constant potential ($\ddot{I} = U + 2h$)
$\Rightarrow$ constant kinetic energy ($h = K - U$)

Key Observation: Both the Smale/Wintner/Chazy question and Saari's conjecture can be formulated using the mutual distances as coordinates. These open questions can be reduced to showing a system of polynomial equations has a finite number of solutions.

The Planar, Circular, Restricted 3-Body Problem (PCR3BP)

$\mathbf{q}_1 = (1-\mu, 0), m_1 = \mu$ and $\mathbf{q}_2 = (-\mu, 0), m_2 = 1-\mu \quad (0 < \mu \leq 1/2)$

Let $a = \sqrt{(x-1+\mu)^2 + y^2}, \quad b = \sqrt{(x+\mu)^2 + y^2}$.

Equations of motion:

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= V_x + 2v \\ \dot{v} &= V_y - 2u \end{aligned}$$

where

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{\mu}{a} + \frac{1-\mu}{b} + \frac{1}{2}\mu(1-\mu)$$

is the **amended potential.**

Jacobi integral: $E = \frac{1}{2}(u^2+v^2) - V \quad \Longrightarrow \quad V(x,y) \geq -E$

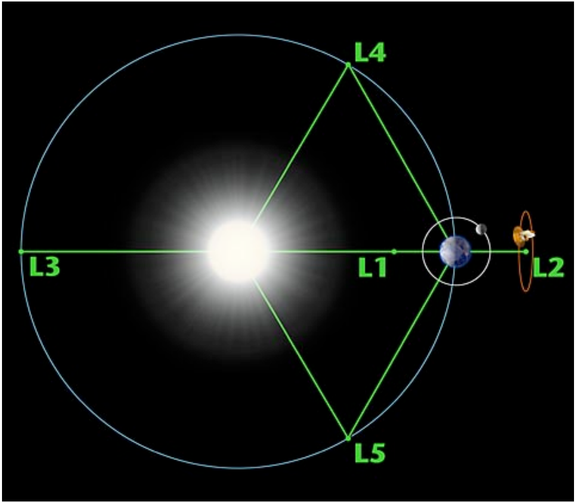


Figure: The five libration points (Lagrange points) in the Sun-Earth system (not drawn to scale).
http://map.gsfc.nasa.gov/mission/observatory_l2.html

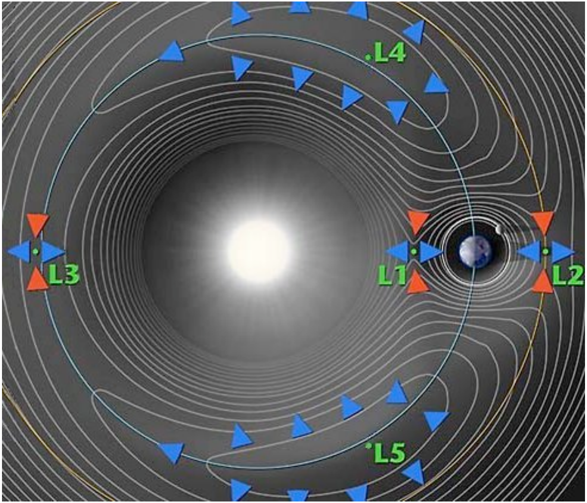


Figure: The level curves for the amended potential and the libration points.
http://map.gsfc.nasa.gov/mission/observatory_l2.html

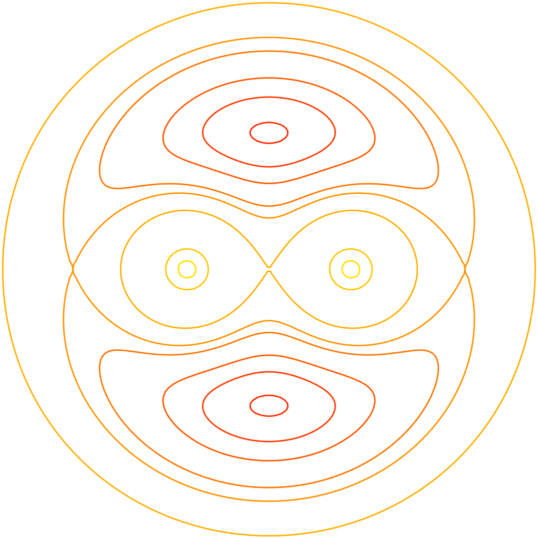


Figure: Level curves of V for $\mu = 1/2$ (equal mass) in the PCR3BP.

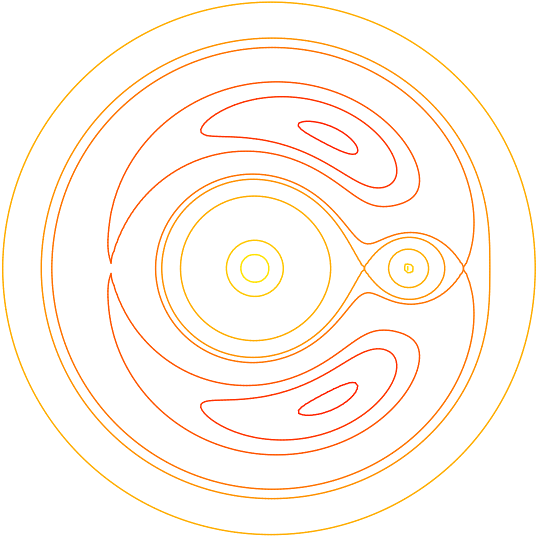


Figure: Level curves of V for $\mu = 0.1$ in the PCR3BP.

Saari's Conjecture Amended to the PCR3BP

Recall: In the full n-body problem, I constant implies both the potential and kinetic energy are also constant.

Theorem

(GR, Melanson 2007) The only solutions to the planar, circular, restricted three-body problem with a constant value of the amended potential V are equilibria (libration points).

Corollary

(GR, Melanson 2007) It is not possible for a solution to the PCR3BP to travel with constant speed without being fixed at one of the libration points.

Proof of Corollary: Due to the Jacobi integral, constant speed implies constant potential V. $\square$

The Equilateral Triangle Solution of Lagrange

Lagrange (1772): Place three bodies, of any masses, at the vertices of an equilateral triangle and apply the appropriate velocities to obtain a relative equilibrium. Each body traces out a circle centered at the center of mass of the triangle. The shape and size of the configuration is preserved during the motion.

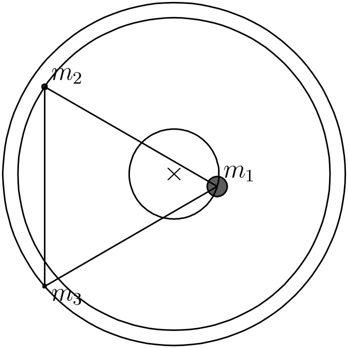


Figure: Lagrange's equilateral triangle solution in the three-body problem.

The Planar, Circular, Restricted Four-Body Problem (PCR4BP)

Insert a fourth infinitesimal mass that has no influence on the circular orbits of the three larger masses ("primaries"). Change to a rotating coordinate system in a frame where the primaries are fixed. Let (x, y) be coordinates for the infinitesimal mass in this new frame.

Equations of motion: (assume $m_1 + m_2 + m_3 = 1$)

$$\begin{aligned} \ddot{x} &= 2\dot{y} + V_x \\ \ddot{y} &= -2\dot{x} + V_y \end{aligned}$$

where

$$V(x, y) = \frac{1}{2}\left((x - c_x)^2 + (y - c_y)^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c}$$

is the **amended potential**, (c_x, c_y) is the center of mass of the primaries and a, b, c represent the respective distances of the infinitesimal mass from each of the three primaries.

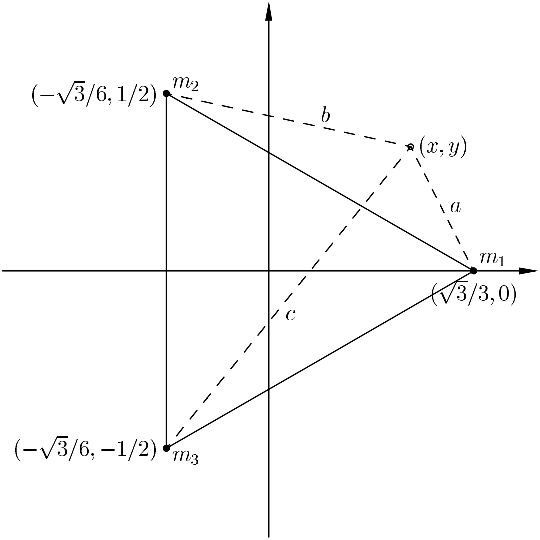


Figure: Setup for the planar, circular, restricted four-body problem.

Two Finiteness Questions

$$V(x,y) = \frac{1}{2}\left((x-c_x)^2 + (y-c_y)^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c}$$

$$\begin{aligned} \ddot{x} &= 2\dot{y} + V_x \\ \ddot{y} &= -2\dot{x} + V_y \end{aligned}$$

Let $\dot{x} = u, \dot{y} = v$. Integral of motion:

$$E = \frac{1}{2}(u^2 + v^2) - V \quad \text{(Jacobi)}$$

Note: Critical points of V are equilibrium points of the PCR4BP ("parking spaces").

1. How do the location and number of critical points change as the masses of the primaries are varied? Are there a **finite** number of critical points for **all** choices of m_1, m_2 and m_3?
2. Is it possible for a solution to the above equations to travel along a level curve of V? (*Saari's Conjecture*)

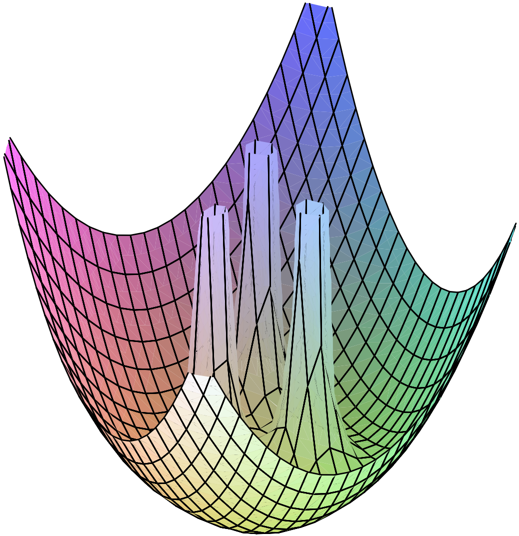


Figure: The amended potential V for the case of three equal masses.

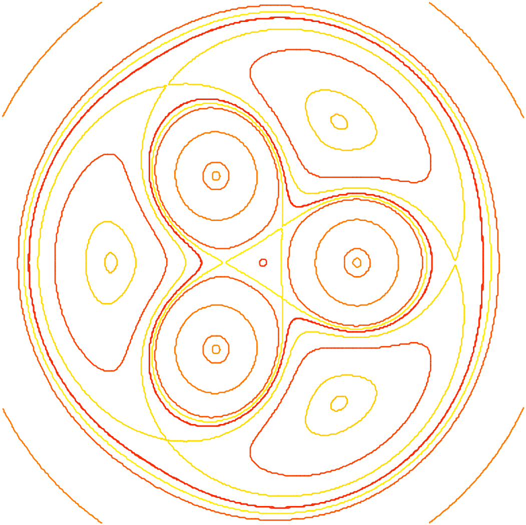


Figure: Level curves of the amended potential V for the case of three equal masses. There are 10 critical points — 6 saddles and 4 minima.

Using Distance Coordinates

Treat the distances a, b, c as variables:

$$x = \frac{\sqrt{3}}{6}\left(b^2 + c^2 - 2a^2\right) \quad y = \frac{1}{2}\left(c^2 - b^2\right)$$

subject to the constraint

$$F = a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) = -1$$

(Cayley-Menger Determinant). In these new coordinates, the amended

potential function becomes

$$V = \frac{1}{2}\left(m_1a^2 + m_2b^2 + m_3c^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c} + \text{constant.}$$

Equations for the Critical Points

$$m_1(1 - \frac{1}{a^3}) + 2\lambda(2a^2 - b^2 - c^2 - 1) = 0 \quad (3)$$

$$m_2(1 - \frac{1}{b^3}) + 2\lambda(2b^2 - a^2 - c^2 - 1) = 0 \quad (4)$$

$$m_3(1 - \frac{1}{c^3}) + 2\lambda(2c^2 - a^2 - b^2 - 1) = 0 \quad (5)$$

$$a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) = -1$$

Summing equations $(1), (2)$ and (3) yields

$$\lambda = \frac{1}{6}\left(1 - \frac{m_1}{a^3} - \frac{m_2}{b^3} - \frac{m_3}{c^3}\right).$$

Eliminating λ

$$2a^5b^3c^3 - 2m_3a^5b^3 - 2m_2a^5c^3 - a^3b^5c^3 + m_3a^3b^5 - a^3b^3c^5$$
$$+(3m_1 - 1)a^3b^3c^3 + m_3a^3b^3c^2 + m_3a^3b^3 + m_2a^3b^2c^3 + m_2a^3c^5$$
$$+m_2a^3c^3 - 2m_1a^2b^3c^3 + m_1b^5c^3 + m_1b^3c^5 - 2m_1b^3c^3 = 0$$

$$2a^3b^5c^3 - 2m_3a^3b^5 - 2m_1b^5c^3 - a^5b^3c^3 + m_3a^5b^3 - a^3b^3c^5$$
$$+(3m_2 - 1)a^3b^3c^3 + m_3a^3b^3c^2 + m_3a^3b^3 + m_1a^2b^3c^3 + m_1b^3c^5$$
$$+m_1b^3c^3 - 2m_2a^3b^2c^3 + m_2a^5c^3 + m_2a^3c^5 - 2m_2a^3c^3 = 0$$

$$a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) + 1 = 0$$

Symmetry: $a \leftrightarrow b, m_1 \leftrightarrow m_2$

Equal Mass Case

Theorem

(GR, JK, CS 2007) The number of critical points in the PCR4BP for equal masses is exactly 10.

Proof: Due to the equal masses, it is possible to show that all critical points must lie on an altitude of the equilateral triangle ($a = b$, $a = c$ or $b = c$). This reduces the problem down to two equations in two unknowns. Using Gröbner bases (or resultants), we obtain a 22 degree polynomial that contains 5 positive real roots. Of these 5, three correspond to physically relevant solutions of the original equations. By symmetry, this gives a total of 9 critical points. The 10th is found at the origin, where all three altitudes intersect.

Remark: This result is subsumed by numerical and analytic work of Pedersen (1944), Simó (1978), Arenstorf (1982) and Leandro (2006).

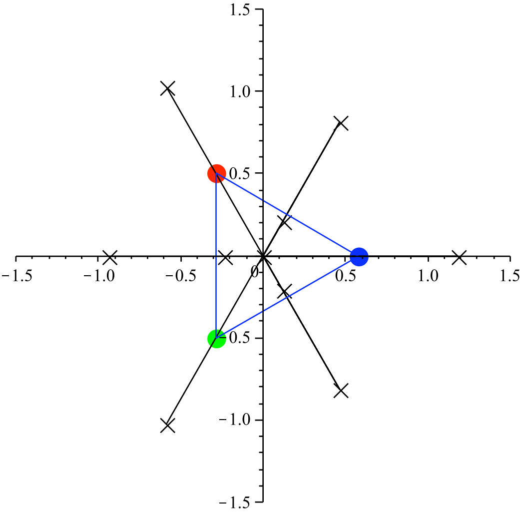


Figure: The 10 equilibria for the PCR4BP in the case of equal masses. Note the symmetry with respect to the equilateral triangle of the primaries.

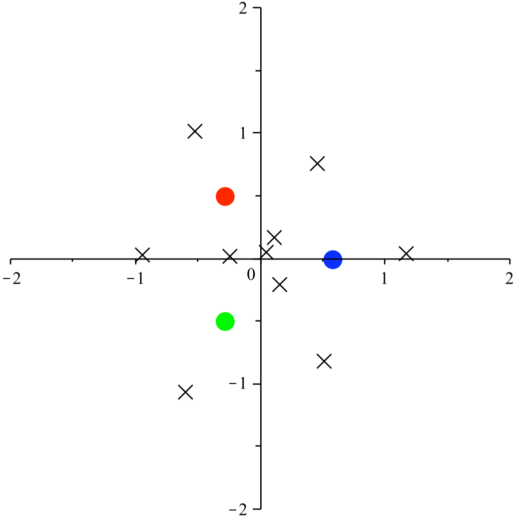


Figure: Equilibria with two equal masses just before a bifurcation.

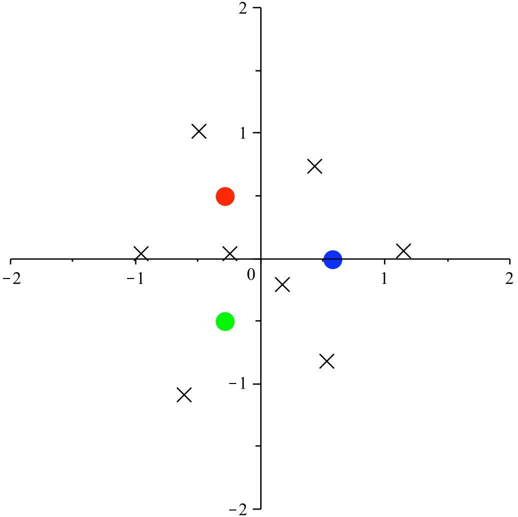


Figure: Equilibria with two equal masses just after a bifurcation.

Theorem

(GR, JK, CS 2007) The number of critical points in the PCR4BP is finite for any choice of masses. In particular, there are less than 268 critical points.

Remark: Our result showing finiteness appears to be new. The upper bound of 268 is not optimal as the work of Pedersen (1944), Simó (1978), Arenstorf (1982) and Leandro (2006) suggests the actual number varies between 8 and 10. It is a surprisingly complicated problem to study the bifurcation curve in the mass parameter space for which there are precisely 9 critical points.

BKK Theory

Bernstein, D. N., The Number of Roots of a System of Equations, *Functional Analysis and its Applications*, **9**, no. 3, 183-185, 1975.

Given $f \in \mathbb{C}[z_1, \ldots z_n]$, $f = \sum c_k z^k$, $\quad k = (k_1, k_2, \ldots, k_n)$, the **Newton polytope** of f, denoted $N(f)$, is the convex hull in $\mathbb{R}^n$ of the set of all exponent vectors occurring for f.

Ex. The constraint equation:

$$a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) + 1 = 0$$

Exponent vectors: $(4,0,0), (0,4,0), (0,0,4), (2,2,0), (2,0,2), (0,2,2)$ $(2,0,0), (0,2,0), (0,0,2), (0,0,0)$

Only need the first three and the last one to describe the convex hull.

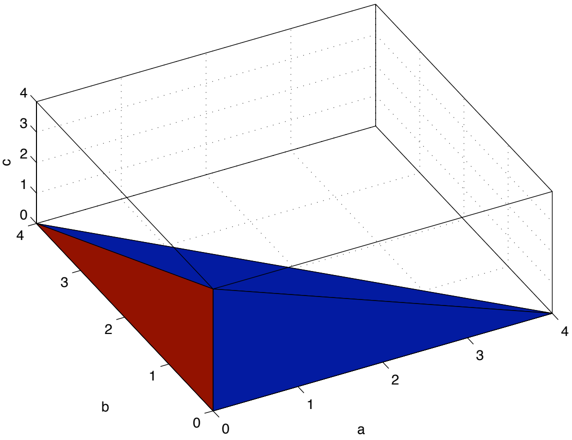


Figure: The Newton polytope for the constraint equation.

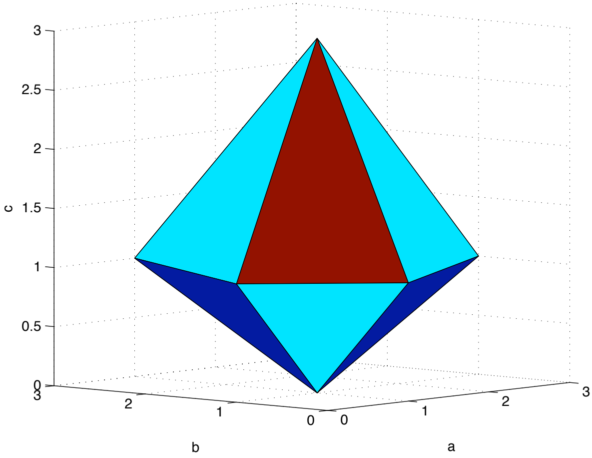


Figure: The Newton polytope for the equation $V = k$ involving the amended potential V.

Reduced Equations

Given $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Q}$, the **reduced polynomial** f_α is the sum of all terms of f whose exponent vectors k satisfy

$$\alpha \cdot k = \min_{l \in N(f)} \alpha \cdot l.$$

This equation defines a face of the polytope $N(f)$ with inward pointing normal α.

Let $\mathbb{T} = (\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$. Since our variables represent distances, we are only concerned with those solutions in $\mathbb{T}$.

Theorem

(Bernstein, 1975) Suppose that system (6) has infinitely many solutions in $\mathbb{T}$. Then there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Q}$ and $\alpha_j = 1$ for some j, such that the system of reduced equations (7) also has a solution in $\mathbb{T}$ (all components nonzero).

$$\begin{aligned} f_1(z_1, \ldots, z_n) &= 0 \\ f_2(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_m(z_1, \ldots, z_n) &= 0, \end{aligned} \tag{6}$$

$$\begin{aligned} f_{1\alpha}(z_1, \ldots, z_n) &= 0 \\ f_{2\alpha}(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_{m\alpha}(z_1, \ldots, z_n) &= 0. \end{aligned} \tag{7}$$

The "big Minkowski"

Key Fact: Bernstein argues that it is sufficient to check a finite number of vectors α since different vectors can induce the **same** reduced equations. Using the Minkowski sum polytope

$$N(f_1) + N(f_2) + \cdots + N(f_m) = \{v \in \mathbb{R}^n : v = v_1 + \cdots + v_m, v_i \in N(f_i)\},$$

only the inward normals of each facet of this "big Minkowski" need be considered. We must also examine the reduced equations for "faces" of codimension greater than one. If all such α's fail to yield a nontrivial solution (all components nonzero), then Bernstein's theorem shows that the number of solutions to the system is finite.

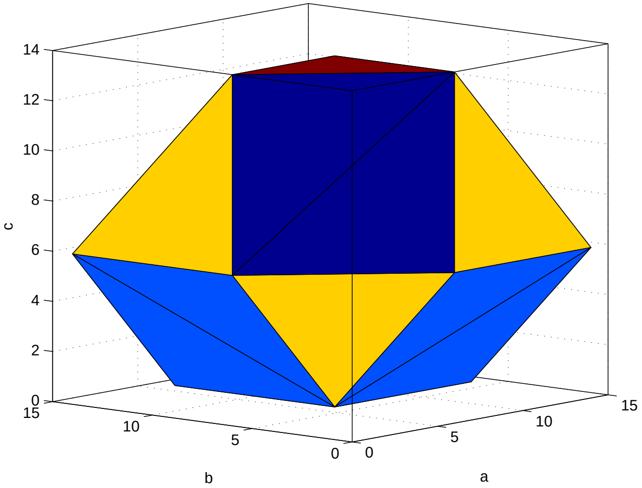


Figure: The Minkowski sum polytope corresponding to the three equations for the critical points of V.

Good Example

Choose $\alpha = < 0, 1, 1 >$

Reduced equations:

$$\begin{aligned}
-2m_3a^5b^3 - 2m_2a^5c^3 + m_3a^3b^3 + m_2a^3c^3 &= 0 \\
m_3a^5b^3 + m_2a^5c^3 + m_3a^3b^3 - 2m_2a^3c^3 &= 0 \\
a^4 - a^2 + 1 &= 0
\end{aligned}$$

Gröbner basis: $\{a^4 - a^2 + 1, m_3b^3, m_2c^3\}$

No solutions in $\mathbb{T}$ means this α is excluded. Yay!

Difficult Example

Choose $\alpha =< 1, 0, 0 >$

Reduced equations:

$$\begin{aligned} m_1 b^5 c^3 + m_1 b^3 c^5 - 2m_1 b^3 c^3 &= 0 \\ -2m_1 b^5 c^3 + m_1 b^3 c^5 + m_1 b^3 c^3 &= 0 \\ b^4 + c^4 - b^2 c^2 - b^2 - c^2 + 1 &= 0 \end{aligned}$$

Problem: $b = \pm 1, c = \pm 1$ and $a \neq 0$ is a nontrivial solution to the reduced equations. Boo! Bernstein's Theorem doesn't help.

Puiseux Series

Hampton, M. and Moeckel, R., Finiteness of relative equilibria of the four-body problem, *Inventiones mathematicae* **163**, 289-312, 2006.

Puiseux series (complex) :

$$x(t) = \sum_{i=i_0}^{\infty} a_i t^{\frac{i}{q}} \,, \quad q \in \mathbb{N}, \; i_0 \in \mathbb{Z}$$

If a system of n polynomial equations has an infinite variety in $\mathbb{T}$, then there exists a convergent Puiseux series solution $x_j(t), j = 1, \ldots n$ with **order** α. Moreover, one of the variables is simply $x_l(t) = t$.

The **order** of the Puiseux series solution is the vector of rationals arising from the fractional exponent of the first term in each series. This vector α is precisely the same α of Bernstein's theorem.

Good News

- Minkowski sum polytope for our system has 14 facets, 12 vertices and 24 edges.

Good News

- Minkowski sum polytope for our system has 14 facets, 12 vertices and 24 edges.
- Using symmetry and ignoring those inward normals with all coordinates non-positive, only two inward normals remain that have reduced equations with nontrivial solutions: $\alpha_1 =< 1, 0, 0 >$ and $\alpha_2 =< 0, 0, 1 >$.

Good News

- Minkowski sum polytope for our system has 14 facets, 12 vertices and 24 edges.
- Using symmetry and ignoring those inward normals with all coordinates non-positive, only two inward normals remain that have reduced equations with nontrivial solutions: $\alpha_1 =< 1, 0, 0 >$ and $\alpha_2 =< 0, 0, 1 >$.
- For each "bad" α, we can substitute Puiseux series in t into the original equations ($a = t$ for α_1 and $c = t$ for α_2), and show that no such series solution can exist by examining higher order terms in t (Implicit Function Theorem).

Good News

- Minkowski sum polytope for our system has 14 facets, 12 vertices and 24 edges.
- Using symmetry and ignoring those inward normals with all coordinates non-positive, only two inward normals remain that have reduced equations with nontrivial solutions: $\alpha_1 =< 1, 0, 0 >$ and $\alpha_2 =< 0, 0, 1 >$.
- For each "bad" α, we can substitute Puiseux series in t into the original equations ($a = t$ for α_1 and $c = t$ for α_2), and show that no such series solution can exist by examining higher order terms in t (Implicit Function Theorem).
- Of the 15 edges that need to be examined, most have reduced equations with either no solution or a trivial solution. The others can be eliminated using symmetry.

Final Remarks and Future Work

- The vertices of the Minkowski sum polytope (faces of codimension 3) are quickly eliminated since they yield at least one reduced equation with a single monomial, and thereby a trivial solution. This completes the proof of finiteness.

Final Remarks and Future Work

- The vertices of the Minkowski sum polytope (faces of codimension 3) are quickly eliminated since they yield at least one reduced equation with a single monomial, and thereby a trivial solution. This completes the proof of finiteness.
- The lower bound of 268 for the number of critical points is obtained by computing the **mixed volume** of the polytopes corresponding to our system of equations (Bernstein).

Final Remarks and Future Work

- The vertices of the Minkowski sum polytope (faces of codimension 3) are quickly eliminated since they yield at least one reduced equation with a single monomial, and thereby a trivial solution. This completes the proof of finiteness.
- The lower bound of 268 for the number of critical points is obtained by computing the **mixed volume** of the polytopes corresponding to our system of equations (Bernstein).
- What about solutions traveling on level curves? Hopefully similar techniques will show this is impossible, but polynomials are much, much larger (over 10,000 terms).

Final Remarks and Future Work

- The vertices of the Minkowski sum polytope (faces of codimension 3) are quickly eliminated since they yield at least one reduced equation with a single monomial, and thereby a trivial solution. This completes the proof of finiteness.
- The lower bound of 268 for the number of critical points is obtained by computing the **mixed volume** of the polytopes corresponding to our system of equations (Bernstein).
- What about solutions traveling on level curves? Hopefully similar techniques will show this is impossible, but polynomials are much, much larger (over 10,000 terms).
- Additional problem: PCRnBP with equal mass primaries on a regular n-gon. Applications to the charged n-body problem?