

# Math and Music: Exploring the Connections

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## Outline

- 1 Music as Science
- 2 Math as Art
- 3 Rhythm
- 4 Symmetry in Music
- 5 Change Ringing

## Some Quotes

*May not Music be described as the Mathematic of Sense, Mathematics as the Music of reason? The soul of each the same! Thus the musician feels Mathematic, the mathematician thinks Music, — Music the dream, Mathematic the working life, — each to receive its consummation from the other.*

*James Joseph Sylvester, 1865*

*Music is the arithmetic of sounds as optics is the geometry of light.*

*Claude Debussy, c. 1900*

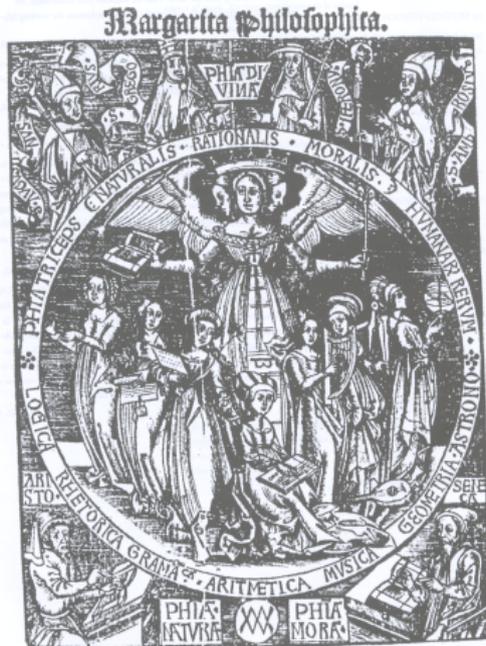


Figure: The *Quadrivium* and the *Trivium*. Woodcut, 1504.

*Quadrivium*: Boethius (c. 480-524) Music, arithmetic, astronomy and geometry.

## Music as Science

### Early British Education

- Music taught as a science, although exams for B. Mus. and D.Mus. required a composition of music
- Gresham Professorships, London, 1596: music, 'physic', geometry and astronomy
- "he is to expound on 'canonics, or music'" (Description of the University Chair in **mathematics** at Oxford University in 1619)
- "Speculative is that kinde of musicke which by Mathematical helps, seeketh out the causes, properties, and natures of soundes." — Thomas Morley, *Plaine and easie introduction to music*, 1597

## Music as Science

*I do present you with a man of mine,  
Cunning in music and in mathematics,  
To instruct her fully in those sciences,  
Whereof, I know, she is not ignorant.  
William Shakespeare, 1594*

- Formalized, internationally accepted notation:

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$



Béla Bartók, *Mikrokosmos*, No. 141, *Subject and reflection*

- **Axiomatic development:** Euclidean Geometry: Definitions and proofs lead to theorems and propositions  
Music Theory: Scales, Harmony, Sequences, Progressions, Sonata-form
- **Theorems and Rules:**  
Mathematics: Pythagorean Theorem, Infinitude of Prime Numbers, Fundamental Theorem of Calculus, Mandelbrot Set is Connected,  $\sqrt{2}$  is irrational  
Music: No parallel fifths or octaves, Dominant leads to the Tonic, Set Theory, Twelve-tone Method
- **The Science of Sound:** Why do some combination of notes sound better than others? The overtone series, Musical acoustics

Equal temperament  $\longleftrightarrow 2^{1/12}$

- **New Journal in 2007:** *Journal of Mathematics and Music:* Mathematical and Computational Approaches to Music Theory, Analysis and Composition

## Math as Art

*But mathematics is the sister, as well as the servant, of the arts and is touched with the same madness and genius.*

*Harold Marston Morse*

- Child Prodigies (Mozart and Gauss)

$$1 + 2 + 3 + \cdots + 98 + 99 + 100 = ?$$

$$101 \cdot 50 = 5050$$

- Emotional, aesthetic connections (eg. Andrew Wiles discussing his proof of Fermat's Last Theorem)
- The Mozart Effect

## Math as Art

### Mathematicians/Musicians are everywhere!

- Edward Teller – physicist, pianist
- Dr. Albert Schweitzer – organist, renowned Bach expert
- Caroline Herschel – astronomer, singer of oratorios
- Donald Knuth – computer scientist, organist, composer
- Albert Einstein – physicist, violinist
- Manjul Bhargava – number theorist, master tabla player (classical Indian music)
- Noam Elkies – wicked smart Harvard mathematician, pianist, composer
- Half of the Holy Cross Math/CS Dept.
- Your Neighbor? Your Department? Your Kids?

## Rhythm: Least Common Multiple

*Music is the pleasure the human soul experiences from counting without being aware that it is counting.*

*Gottfried Leibniz*

Counting a 2-against-3  
rhythmic cycle

2 notes / measure  $\times$  3 counts / note = 6 counts / measure

3 notes / measure  $\times$  2 counts / note = 6 counts / measure

1 and 2 and 3 and

$$\text{lcm}(2, 3) = 6$$

## Rhythm: Least Common Multiple

Counting a 3-against-4  
rhythmic cycle

4 notes / measure  $\times$  3 counts / note = 12 counts / measure

3 notes / measure  $\times$  4 counts / note = 12 counts / measure

1 e and a 2 e and a 3 e and a

$$\text{lcm}(3, 4) = 12$$

# Rhythm: Least Common Multiple

## Music of Chopin

22-against-12

A musical score for a piano piece. The top staff is in treble clef and contains a melodic line with a slur over 22 notes. The bottom staff is in bass clef and contains a bass line with a slur over 12 notes. The key signature has three flats (B-flat, E-flat, A-flat).

11-against-6

A musical score for a piano piece. The top staff is in treble clef and contains a melodic line with a slur over 11 notes. The bottom staff is in bass clef and contains a bass line with a slur over 6 notes. The key signature has three flats (B-flat, E-flat, A-flat). A box highlights the 11-note melodic phrase and the 6-note bass phrase.

# Rhythm: Least Common Multiple

## Music of Chopin

The image shows a musical score for piano with two staves. The key signature has three sharps (F#, C#, G#). The score is divided into two sections. The first section is labeled "8-against-3" and shows an 8-measure phrase in the treble clef and a 3-measure phrase in the bass clef. The second section is labeled "5-against-3" and shows a 5-measure phrase in the treble clef and a 3-measure phrase in the bass clef. Brackets and numbers (8, 3, 5, 5) are used to group the notes in each part to show the polyrhythmic relationship.

$$\text{lcm}(a, b) = \frac{ab}{\text{gcd}(ab)}$$

## Polyrhythmic Music

- Multiple rhythms at once
- African music often polyrhythmic with different drums and percussion instruments playing different rhythms simultaneously
- Classical Indian Music: **tabla** (pair of small hand drums). Drummers often play challenging combinations such as 11 beats in one hand and 12 in the other. Prime numbers play an important role.



Figure: A tabla

Requiem Mass, Giuseppe Verdi (No. 2 Dies irae) 41

The image shows a page of a musical score for Giuseppe Verdi's Requiem, specifically the Dies irae section. The score is divided into three systems. The first system (measures 110-112) features vocal lines with lyrics: "cum vix je-ctus ait so - cu - ras? quem pa - truum ro - ga - tu - ras? mi - ser lan - di - ctu - ras?". The second system (measures 113-114) is marked "Adagio maestoso (d. 72)" and includes a vocal line with lyrics: "Rex tre - men - dena - je - ctu - tis, Rex tremen - da - je - ctu - tis, rex tremen - da - je - ctu - tis." and a piano accompaniment. The third system (measures 115-116) is also marked "Adagio maestoso (d. 72)" and includes a vocal line with lyrics: "Sal - va - san - ctus pi - e - ta - tis, qui sal - van - dos nul - los gre - git, qui sal - van - dos nul - los gre - git." and a piano accompaniment. A handwritten note "Triple Dot!" is written above the piano accompaniment in the second system. The score includes various musical notations such as dynamics (pp, ppp, ff), articulation (accents), and performance instructions (Soprano, Tenor). The publisher's name "Edition Peters" and the number "10943" are visible at the bottom.

Figure: A triple dot in the Verdi Requiem.



## Rhythm: Geometric Series

Quadruple Dot  $\iff$  Geometric Series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16} \text{ beats}$$

$$a_0 + a_0r + a_0r^2 + \cdots + a_0r^{n-1} = \frac{a_0(1 - r^n)}{1 - r}$$

What length does an **infintely** dotted quarter note get?

**Answer:** 2 beats or 1 half note (infinite geometric series)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - 1/2} = 2$$

## Symmetry in Music: Group Theory

How to get more music out of a little motif:

Translations (shifting graph vertically)  $\iff$  Transpositions (shifting notes up or down)

Ex: **Ballpark Music**

Vertical Reflection (symmetry between right and left)  $\iff$   
Retrograde (music same forward and backward)

Ex: **Lean on Me**

Horizontal Reflection (symmetry between top and bottom)  $\iff$   
Inversion (what goes up, must come down)

Ex: **Bach, Bach and more Bach**

# Symmetry in Music: Retrograde

MENUET AL ROVESCIO

The image displays a musical score for a minuet titled "Menuet al Rovescio" (Minuet in Reverse). The score is written for piano and is in 3/4 time, with a key signature of one sharp (F#). The title "MENUET AL ROVESCIO" is centered above the first system. The score consists of four systems of music, each with a treble and bass clef staff. The first system shows the beginning of the piece. The second system continues the melody. The third system shows the end of the piece, marked with a double bar line and repeat dots. The fourth system shows the beginning of the piece again, demonstrating the retrograde symmetry. The music is a simple, elegant minuet.

Figure: Joseph Haydn, *Piano Sonata No. 41, Hob. XVI/26*, Minuet



## Symmetry in Music: Retrograde

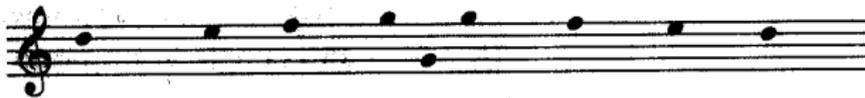
The image displays five systems of musical notation for Johann Sebastian Bach's *A Musical Offering*. Each system consists of two staves: a treble clef staff on top and a bass clef staff on the bottom. The music is written in G major (one sharp) and 3/4 time. The first system shows the beginning of the piece. The second system continues the melody. The third system features a more complex rhythmic pattern with sixteenth notes. The fourth system continues with similar rhythmic complexity. The fifth system concludes the piece with a final cadence. The notation includes various note values (quarter, eighth, sixteenth notes), rests, and bar lines, illustrating the intricate structure of the piece.

Figure: Johann Sebastian Bach, *A Musical Offering*

## Symmetry in Music



for the Lord God Om - ni - po - tent reign - eth



George F. Handel, *Messiah*, *Hallelujah chorus* (lose retrograde, form of tone painting)

**Allegro**

Musical notation for the beginning of Mikrokosmos No. 141. It is a piano piece in 3/4 time, marked 'Allegro'. The notation is for the first system, showing the right and left hands. The right hand starts with a treble clef and the left hand with a bass clef. The key signature has two flats (B-flat and E-flat). The melody in the right hand is a sequence of eighth notes: B4, A4, G4, F#4, E4, D4, C4, B3, A3, G3, F#3, E3, D3, C3, B2, A2, G2, F#2, E2, D2, C2, B1, A1, G1, F#1, E1, D1, C1, B0, A0, G0, F#0, E0, D0, C0, B-1, A-1, G-1, F#-1, E-1, D-1, C-1, B-2, A-2, G-2, F#-2, E-2, D-2, C-2, B-3, A-3, G-3, F#-3, E-3, D-3, C-3, B-4, A-4, G-4, F#-4, E-4, D-4, C-4, B-5, A-5, G-5, F#-5, E-5, D-5, C-5, B-6, A-6, G-6, F#-6, E-6, D-6, C-6, B-7, A-7, G-7, F#-7, E-7, D-7, C-7, B-8, A-8, G-8, F#-8, E-8, D-8, C-8, B-9, A-9, G-9, F#-9, E-9, D-9, C-9, B-10, A-10, G-10, F#-10, E-10, D-10, C-10, B-11, A-11, G-11, F#-11, E-11, D-11, C-11, B-12, A-12, G-12, F#-12, E-12, D-12, C-12, B-13, A-13, G-13, F#-13, E-13, D-13, C-13, B-14, A-14, G-14, F#-14, E-14, D-14, C-14, B-15, A-15, G-15, F#-15, E-15, D-15, C-15, B-16, A-16, G-16, F#-16, E-16, D-16, C-16, B-17, A-17, G-17, F#-17, E-17, D-17, C-17, B-18, A-18, G-18, F#-18, E-18, D-18, C-18, B-19, A-19, G-19, F#-19, E-19, D-19, C-19, B-20, A-20, G-20, F#-20, E-20, D-20, C-20, B-21, A-21, G-21, F#-21, E-21, D-21, C-21, B-22, A-22, G-22, F#-22, E-22, D-22, C-22, B-23, A-23, G-23, F#-23, E-23, D-23, C-23, B-24, A-24, G-24, F#-24, E-24, D-24, C-24, B-25, A-25, G-25, F#-25, E-25, D-25, C-25, B-26, A-26, G-26, F#-26, E-26, D-26, C-26, B-27, A-27, G-27, F#-27, E-27, D-27, C-27, B-28, A-28, G-28, F#-28, E-28, D-28, C-28, B-29, A-29, G-29, F#-29, E-29, D-29, C-29, B-30, A-30, G-30, F#-30, E-30, D-30, C-30, B-31, A-31, G-31, F#-31, E-31, D-31, C-31, B-32, A-32, G-32, F#-32, E-32, D-32, C-32, B-33, A-33, G-33, F#-33, E-33, D-33, C-33, B-34, A-34, G-34, F#-34, E-34, D-34, C-34, B-35, A-35, G-35, F#-35, E-35, D-35, C-35, B-36, A-36, G-36, F#-36, E-36, D-36, C-36, B-37, A-37, G-37, F#-37, E-37, D-37, C-37, B-38, A-38, G-38, F#-38, E-38, D-38, C-38, B-39, A-39, G-39, F#-39, E-39, D-39, C-39, B-40, A-40, G-40, F#-40, E-40, D-40, C-40, B-41, A-41, G-41, F#-41, E-41, D-41, C-41, B-42, A-42, G-42, F#-42, E-42, D-42, C-42, B-43, A-43, G-43, F#-43, E-43, D-43, C-43, B-44, 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C-151, B-152, A-152, G-152, F#-152, E-152, D-152, C-152, B-153, A-153, G-153, F#-153, E-153, D-153, C-153, B-154, A-154, G-154, F#-154, E-154, D-154, C-154, B-155, A-155, G-155, F#-155, E-155, D-155, C-155, B-156, A-156, G-156, F#-156, E-156, D-156, C-156, B-157, A-157, G-157, F#-157, E-157, D-157, C-157, B-158, A-158, G-158, F#-158, E-158, D-158, C-158, B-159, A-159, G-159, F#-159, E-159, D-159, C-159, B-160, A-160, G-160, F#-160, E-160, D-160, C-160, B-161, A-161, G-161, F#-161, E-161, D-161, C-161, B-162, A-162, G-162, F#-162, E-162, D-162, C-162, B-163, A-163, G-163, F#-163, E-163, D-163, C-163, B-164, A-164, G-164, F#-164, E-164, D-164, C-164, B-165, A-165, G-165, F#-165, E-165, D-165, C-165, B-166, A-166, G-166, F#-166, E-166, D-166, C-166, B-167, A-167, G-167, F#-167, E-167, D-167, C-167, B-168, A-168, G-168, F#-168, E-168, D-168, C-168, B-169, A-169, G-169, F#-169, E-169, D-169, C-169, B-170, A-170, G-170, F#-170, E-170, D-170, C-170, B-171, A-171, G-171, F#-171, E-171, D-171, C-171, B-172, A-172, G-172, F#-172, E-172, D-172, C-172, B-173, A-173, G-173, F#-173, E-173, D-173, C-173, B-174, A-174, G-174, F#-174, E-174, D-174, C-174, B-175, A-175, G-175, F#-175, E-175, D-175, C-175, B-176, A-176, G-176, F#-176, E-176, D-176, C-176, B-177, A-177, G-177, F#-177, E-177, D-177, C-177, B-178, A-178, G-178, F#-178, E-178, D-178, C-178, B-179, A-179, G-179, F#-179, E-179, D-179, C-179, B-180, A-180, G-180, F#-180, E-180, D-180, C-180, B-181, A-181, G-181, F#-181, E-181, D-181, C-181, B-182, A-182, G-182, F#-182, E-182, D-182, C-182, B-183, A-183, G-183, F#-183, E-183, D-183, C-183, B-184, A-184, G-184, F#-184, E-184, D-184, C-184, B-185, A-185, G-185, F#-185, E-185, D-185, C-185, B-186, A-186, G-186, F#-186, E-186, D-186, C-186, B-187, A-187, G-187, F#-187, E-187, D-187, C-187, B-188, A-188, G-188, F#-188, E-188, D-188, C-188, B-189, A-189, G-189, F#-189, E-189, D-189, C-189, B-190, A-190, G-190, F#-190, E-190, D-190, C-190, B-191, A-191, G-191, F#-191, E-191, D-191, C-191, B-192, A-192, G-192, F#-192, E-192, D-192, C-192, B-193, A-193, G-193, F#-193, E-193, D-193, C-193, B-194, A-194, G-194, F#-194, E-194, D-194, C-194, B-195, A-195, G-195, F#-195, E-195, D-195, C-195, B-196, A-196, G-196, F#-196, E-196, D-196, C-196, B-197, A-197, G-197, F#-197, E-197, D-197, C-197, B-198, A-198, G-198, F#-198, E-198, D-198, C-198, B-199, A-199, G-199, F#-199, E-199, D-199, C-199, B-200, A-200, G-200, F#-200, E-200, D-200, C-200, B-201, A-201, G-201, F#-201, E-201, D-201, C-201, B-202, A-202, G-202, F#-202, E-202, D-202, C-202, B-203, A-203, G-203, F#-203, E-203, D-203, C-203, B-204, A-204, G-204, F#-204, E-204, D-204, C-204, B-205, A-205, G-205, F#-205, E-205, D-205, C-205, B-206, A-206, G-206, F#-206, E-206, D-206, C-206, B-207, A-207, G-207, F#-207, E-207, D-207, C-207, B-208, A-208, G-208, F#-208, E-208, D-208, C-208, B-209, A-209, G-209, F#-209, E-209, D-209, C-209, B-210, A-210, G-210, F#-210, E-210, D-210, C-210, B-211, A-211, G-211, F#-211, E-211, D-211, C-211, B-212, A-212, G-212, F#-212, E-212, D-212, C-212, B-213, A-213, G-213, F#-213, E-213, D-213, C-213, B-214, A-214, G-214, F#-214, E-214, D-214, C-214, B-215, A-215, G-215, F#-215, E-215, D-215, C-215, B-216, A-216, G-216, F#-216, E-216, D-216, C-216, B-217, A-217, G-217, F#-217, E-217, D-217, C-217, B-218, A-218, G-218, F#-218, E-218, D-218, C-218, B-219, A-219, G-219, F#-219, E-219, D-219, C-219, B-220, A-220, G-220, F#-220, E-220, D-220, C-220, B-221, A-221, G-221, F#-221, E-221, D-221, C-221, B-222, A-222, G-222, F#-222, E-222, D-222, C-222, B-223, A-223, G-223, F#-223, E-223, D-223, C-223, B-224, A-224, G-224, F#-224, E-224, D-224, C-224, B-225, A-225, G-225, F#-225, E-225, D-225, C-225, B-226, A-226, G-226, F#-226, E-226, D-226, C-226, B-227, A-227, G-227, F#-227, E-227, D-227, C-227, B-228, A-228, G-228, F#-228, E-228, D-228, C-228, B-229, A-229, G-229, F#-229, E-229, D-229, C-229, B-230, A-230, G-230, F#-230, E-230, D-230, C-230, B-231, A-231, G-231, F#-231, E-231, D-231, C-231, B-232, A-232, G-232, F#-232, E-232, D-232, C-232, B-233, A-233, G-233, F#-233, E-233, D-233, C-233, B-234, A-234, G-234, F#-234, E-234, D-234, C-234, B-235, A-235, G-235, F#-235, E-235, D-235, C-235, B-236, A-236, G-236, F#-236, E-236, D-236, C-236, B-237, A-237, G-237, F#-237, E-237, D-237, C-237, B-238, A-238, G-238, F#-238, E-238, D-238, C-238, B-239, A-239, G-239, F#-239, E-239, D-239, C-239, B-240, A-240, G-240, F#-240, E-240, D-240, C-240, B-241, A-241, G-241, F#-241, E-241, D-241, C-241, B-242, A-242, G-242, F#-242, E-242, D-242, C-242, B-243, A-243, G-243, F#-243, E-243, D-243, C-243, B-244, A-244, G-244, F#-244, E-244, D-244, C-244, B-245, A-245, G-245, F#-245, E-245, D-245, C-245, B-246, A-246, G-246, F#-246, E-246, D-246, C-246, B-247, A-247, G-247, F#-247, E-247, D-247, C-247, B-248, A-248, G-248, F#-248, E-248, D-248, C-248, B-249, A-249, G-249, F#-249, E-249, D-249, C-249, B-250, A-250, G-250, F#-250, E-250, D-250, C-250, B-251, A-251, G-251, F#-251, E-251, D-251, C-251, B-252, A-252, G-252, F#-252, E-252, D-252, C-252, B-253, A-253, G-253, F#-253, E-253, D-253, C-253, B-254, A-254, G-254, F#-254, E-254, D-254, C-254, B-255, A-255, G-255, F#-255, E-255, D-255, C-255, B-256, A-256, G-256, F#-256, E-256, D-256, C-256, B-257, A-257, G-257, F#-257, E-257, D-257, C-257, B-258, A-258, G-258, F#-258, E-258, D-258, C-258, B-259, A-259, G-259, F#-259, E-259, D-259, C-259, B-260, A-260, G-260, F#-260, E-260, D-260, C-260, B-261, A-261, G-261, F#-261, E-261, D-261, C-261, B-262, A-262, G-262, F#-262, E-262, D-262, C-262, B-263, A-263, G-263, F#-263, E-263, D-263, C-263, B-264, A-264, G-264, F#-264, E-264, D-264, C-264, B-265, A-265, G-265, F#-265, E-265, D-265, C-265, B-266, A-266, G-266, F#-266, E-266, D-266, C-266, B-267, A-267, G-267, F#-267, E-267, D-267, C-267, B-268, A-268, G-268, F#-268, E-268, D-268, C-268, B-269, A-269, G-269, F#-269, E-269, D-269, C-269, B-270, A-270, G-270, F#-270, E-270, D-270, C-270, B-271, A-271, G-271, F#-271, E-271, D-271, C-271, B-272, A-272, G-272, F#-272, E-272, D-272, C-272, B-273, A-273, G-273, F#-273, E-273, D-273, C-273, B-274, A-274, G-274, F#-274, E-274, D-274, C-274, B-275, A-275, G-275, F#-275, E-275, D-275, C-275, B-276, A-276, G-276, F#-276, E-276, D-276, C-276, B-277, A-277, G-277, F#-277, E-277, D-277, C-277, B-278, A-278, G-278, F#-278, E-278, D-278, C-278, B-279, A-279, G-279, F#-279, E-279, D-279, C-279, B-280, A-280, G-280, F#-280, E-280, D-280, C-280, B-281, A-281, G-281, F#-281, E-281, D-281, C-281, B-282, A-282, G-282, F#-282, E-282, D-282, C-282, B-283, A-283, G-283, F#-283, E-283, D-283, C-283, B-284, A-284, G-284, F#-284, E-284, D-284, C-284, B-285, A-285, G-285, F#-285, E-285, D-285, C-285, B-286, A-286, G-286, F#-286, E-286, D-286, C-286, B-287, A-287, G-287, F#-287, E-287, D-287, C-287, B-288, A-288, G-288, F#-288, E-288, D-288, C-288, B-289, A-289, G-289, F#-289, E-289, D-289, C-289, B-290, A-290, G-

## Symmetry in Music: Rotation



—and here follows an hour of music—



Figure: Paul Hindemith, *Ludus tonalis*, beginning and end.

# Combining Symmetries

9a.

1 2 3 4    2 2 3 4    3 2 3 4    4 2 3 4

Bb    Bb6    Cm7    F7    Bb    E<sup>o</sup>7    Cm7

**"I Got Rhythm" has an AABA structure, and a two-bar tag at the end.**

We call these four equal sections A, A, B, and A. Three of the sections are the same, and one is different. Listen to the first section, and you'll be able

9b.

[A] Bb Bb6 Cm7 F7 Bb6 E<sup>o</sup>7 Cm7 F7 Bb Bb6 Cm7 F7 Ebm6

[A] Bb F7 Bb Bb6 Cm7 F7 Bb6 E<sup>o</sup>7 Cm7 F7 Bb Bb6

[B] Bridge Cm7 F7 Ebm6 Bb F7 Bb D7 Am7 Fm6 D7 G D+ G9 G7

[A] C7 Gm7 Ebm6 C9 C7-5 F7 Bb Bb6 Cm7 F7 Bb6 E<sup>o</sup>7

Tag Cm7 F7 Bb Bb6 Cm7 F7 Ebm Bb Fm G7 C7 F7 Bb

**Figure:** George Gershwin, *I Got Rhythm*, (transposition, retrograde and inversion, all in one song!)

The image displays a musical score for Arnold Schoenberg's *Piano Suite, Op. 25*, specifically the opening of the Minuet and Trio sections. The score is written for piano and is divided into two main sections, A and B.

**Section A (TRIO A):** This section begins with a *forte* (*f*) dynamic and a *martellato* (hammered) articulation. The music is in 3/4 time and features complex rhythmic patterns, including sixteenth and thirty-second notes. A first ending bracket is present, leading to a second ending. The score includes performance instructions such as *f*, *mf*, and *ff*.

**Section B:** This section starts with a *piano* (*p*) dynamic and a *poco più* (slightly more) tempo marking. It features intricate rhythmic figures and dynamic contrasts, including *f*, *mp*, and *p*. The score includes performance instructions such as *poco più*, *f*, *mp*, and *p*. A note is marked with an asterisk and the instruction *\* order inverted*.

The score is written in a system of two staves (treble and bass clef) and includes various musical notations such as dynamics, articulations, and performance markings.

Figure: Arnold Schoenberg, *Piano Suite, Op. 25*, opening of the Minuet and Trio

# Symmetry in Music: Twelve-tone Method

The figure displays four musical staves, each representing a different tone row and its symmetrical counterpart. The notes are numbered 1 through 12, corresponding to the chromatic scale. The first staff, labeled P-0, shows a tone row with a circled interval between notes 3 and 4. The second staff, labeled P-6, shows the same tone row with a circled interval between notes 3 and 4, and an arrow labeled R-6 pointing to the right. The third staff, labeled I-0, shows the same tone row with a circled interval between notes 3 and 4. The fourth staff, labeled I-6, shows the same tone row with a circled interval between notes 3 and 4, and an arrow labeled RI-6 pointing to the left.

Figure: Tone Rows and their symmetries

## Change Ringing (Bell Ringing)

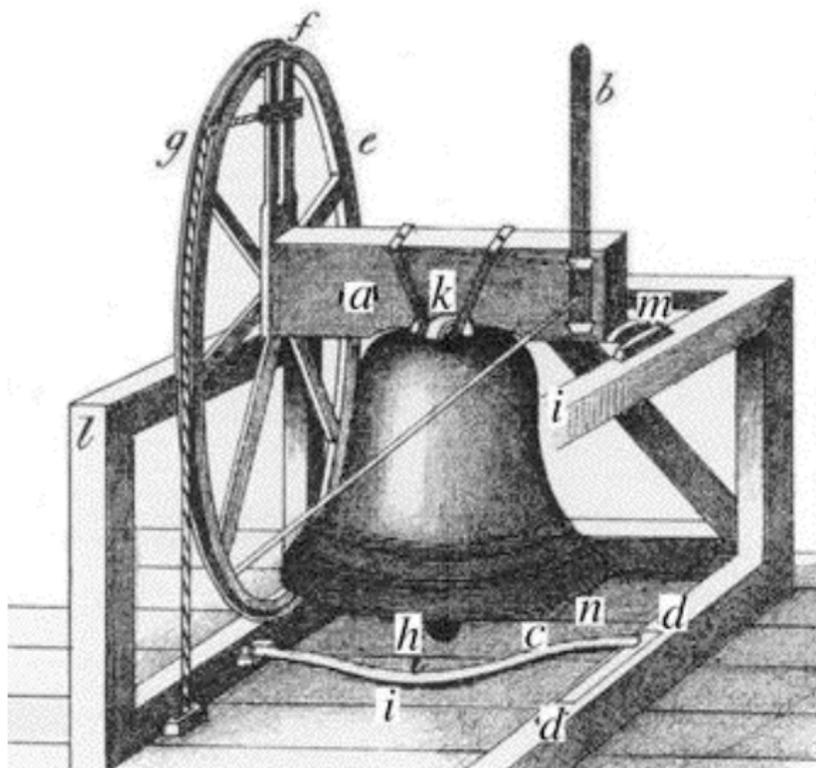


Figure: A typical church bell rung in the belfry.



**Figure:** Bell ringing practice in Stoke Gabriel parish church, south Devon, England.

*Change ringing is a non-competitive and non-violent team activity that is highly stimulating intellectually and mildly demanding physically, and makes a beautiful sound. It develops mental and physical skills in a context of communal effort. The intense concentration required brings euphoric detachment that cleanses the mind of the day's petty demands and frustrations.*

*North American Guild of Change Ringers*

## Change Ringing: An Example

1 2 3 4  
2 1 4 3  
2 4 1 3  
2 4 3 1  
4 2 3 1  
4 2 1 3  
4 1 2 3  
1 4 3 2

1 3 4 2  
3 1 2 4  
3 2 1 4  
3 2 4 1  
2 3 4 1  
2 3 1 4  
2 1 3 4  
1 2 4 3

1 4 2 3  
4 1 3 2  
4 3 1 2  
4 3 2 1  
3 4 2 1  
3 4 1 2  
3 1 4 2  
1 3 2 4  
1 2 3 4

Canterbury Minimus (true extent on 4 bells)

There are  $4! = 24$  different possible rows. Each must be rung exactly once starting and ending with rounds (1 2 3 4).

## Change Ringing: Rules

Rules to ring an **extent** on  $n$  bells:

- 1 The first and last changes (rows) are rounds (1 2 3 4  $\dots$   $n$ ).
- 2 Other than rounds, all of the other  $n!$  changes occur exactly once.
- 3 Between successive changes, no bell moves more than one position.
- 4 No bell rests for more than 2 (sometimes relaxed further to 4) positions.
- 5 Each working bell should do the same amount of “work” (obey the same overall pattern).
- 6 Horizontal symmetry should be present in the extent to help the ringers learn the path of their respective bell. This is called the *palindrome property*.

**Note:** Rules 1 - 3 are mandatory for an extent while Rules 4 - 6 are optional though often satisfied.

## Change Ringing and Mathematics

- A reordering of the numbers  $1\ 2\ 3\ 4\ \dots\ n$  is called a **permutation**.
- How many possible changes on  $n$  bells?  
**Answer:**  $n!$
- For  $n$  bells, how many "moves" are allowed?

$$n = 2 \quad (12) \quad 1 \text{ move}$$

$$n = 3 \quad (12), (23) \quad 2 \text{ moves}$$

$$n = 4 \quad (12), (23), (34), (12)(34) \quad 4 \text{ moves}$$

$$n = 5 \quad (12), (23), (34), (45), (12)(34), (12)(45), (23)(45) \quad 7 \text{ moves!}$$

## Change Ringing and Mathematics

$$n = 6 \quad (12), (23), (34), (45), (56)$$

$$(12)(34), (12)(45), (12)(56), (23)(45), (23)(56), (34)(56)$$

$$(12)(34)(56) \quad 12 \text{ moves}$$

What's the pattern?

$$1, 2, 4, 7, 12 \dots$$

Add one to our sequence:

$$2, 3, 5, 8, 13 \dots$$

### The Fibonacci Sequence!

The number of allowable moves on  $n$  bells is  $F_{n+1} - 1$ .

$n$	$n!$	Approximate Duration	Name
3	6	15 secs.	<i>Singles</i>
4	24	1 mins.	<i>Minimus</i>
5	120	5 mins.	<i>Doubles</i>
6	720	30 mins.	<i>Minor</i>
7	5,040	3 hrs.	<i>Triples</i>
8	40,320	24 hrs.	<i>Major</i>
9	362,880	9 days	<i>Caters</i>
10	3,628,800	3 months	<i>Royal</i>
11	39,916,800	3 years	<i>Cinques</i>
12	479,001,600	36 years	<i>Maximus</i>

**Table:** Approximate duration to ring an extent on  $n$  bells and the names given to such an extent. Compositions: *Plain Bob Minimus*, *Grandshire Triples*

## Change Ringing: 3 bells

### The two extents on 3 bells:

1 2 3	1 2 3
2 1 3	1 3 2
2 3 1	3 1 2
3 2 1	3 2 1
3 1 2	2 3 1
1 3 2	2 1 3
1 2 3	1 2 3

Note the simple zig-zag pattern of Bell **1** in the first extent, sweeping easily from position 1 to position 3 and back again. We say that Bell **1** is *plain hunting*. It only needs to do this once to complete the extent. In this case, we say that the bell is “not working.” Notice that in the second extent, Bell **1** follows a similar zig-zag path except that this begins on the second change.

## Change Ringing

**Plain Bob Minimus** (read down first, then hop to next column)

1 2 3 4  
2 1 4 3  
2 4 1 3  
4 2 3 1  
4 3 2 1  
3 4 1 2  
3 1 4 2  
1 3 2 4

1 3 4 2  
3 1 2 4  
3 2 1 4  
2 3 4 1  
2 4 3 1  
4 2 1 3  
4 1 2 3  
1 4 3 2

1 4 2 3  
4 1 3 2  
4 3 1 2  
3 4 2 1  
3 2 4 1  
2 3 1 4  
2 1 3 4  
1 2 4 3  
1 2 3 4

Let  $a = (12)(34)$ ,  $b = (23)$ ,  $c = (34)$ . The above sequence of 24 permutations can be "factored" as

$$[(ab)^3 ac]^3 = [abababa c]^3 \quad \text{Palindrome!}$$