Math and Music: Exploring the Connections

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Outline

1. Music as Science
2. Math as Art
3. Rhythm
4. Symmetry in Music
5. Change Ringing

Some Quotes

May not Music be described as the Mathematic of Sense, Mathematics as the Music of reason? The soul of each the same! Thus the musician feels Mathematic, the mathematician thinks Music, — Music the dream, Mathematic the working life, — each to receive its consummation from the other.
James Joseph Sylvester, 1865

Music is the arithmetic of sounds as optics is the geometry of light.
Claude Debussy, c. 1900

Figure: The *Quadrivium* and the *Trivium*. Woodcut, 1504.

Quadrivium: Boethius (c. 480-524) Music, arithmetic, astronomy and geometry.

Music as Science

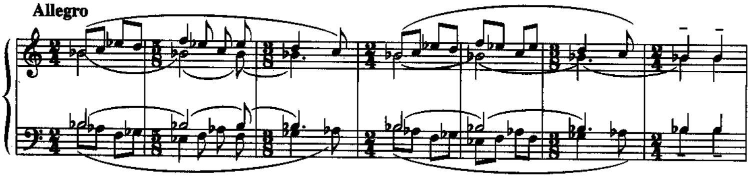
Early British Education

- Music taught as a science, although exams for B. Mus. and D.Mus. required a composition of music
- Gresham Professorships, London, 1596: music, 'physic', geometry and astronomy
- "he is to expound on 'canonics, or music'" (Description of the University Chair in mathematics at Oxford University in 1619)
- "Speculative is that kinde of musicke which by Mathematical helpes, seeketh out the causes, properties, and natures of soundes." — Thomas Morley, *Plaine and easie introduction to music*, 1597

I do present you with a man of mine,
Cunning in music and in mathematics,
To instruct her fully in those sciences,
Whereof, I know, she is not ignorant.
William Shakespeare, 1594

- Formalized, internationally accepted notation:

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$



Béla Bartók, *Mikrokosmos, No. 141, Subject and reflection*

- Axiomatic development: Euclidean Geometry: Definitions and proofs lead to theorems and propositions
  Music Theory: Scales, Harmony, Sequences, Progressions, Sonata-form
- Theorems and Rules:
  Mathematics: Pythagorean Theorem, Infinitude of Prime Numbers, Fundamental Theorem of Calculus, Mandelbrot Set is Connected, $\sqrt{2}$ is irrational
  Music: No parallel fifths or octaves, Dominant leads to the Tonic, Set Theory, Twelve-tone Method
- The Science of Sound: Why do some combination of notes sound better than others? The overtone series, Musical acoustics

$$\text{Equal temperament} \quad \longleftrightarrow \quad 2^{1/12}$$

- New Journal in 2007: *Journal of Mathematics and Music*: Mathematical and Computational Approaches to Music Theory, Analysis and Composition

Math as Art

But mathematics is the sister, as well as the servant, of the arts and is touched with the same madness and genius.
Harold Marston Morse

- Child Prodigies (Mozart and Gauss)

$$1 + 2 + 3 + \cdots + 98 + 99 + 100 \; = ?$$

$$101 \cdot 50 \; = \; 5050$$

- Emotional, aesthetic connections (eg. Andrew Wiles discussing his proof of Fermat's Last Theorem)
- The Mozart Effect

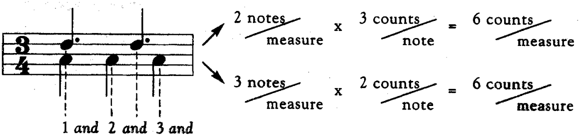
Math as Art

Mathematicians/Musicians are everywhere!

- Edward Teller – physicist, pianist
- Dr. Albert Schweitzer – organist, renowned Bach expert
- Caroline Herschel – astronomer, singer of oratorios
- Donald Knuth – computer scientist, organist, composer
- Albert Einstein – physicist, violinist
- Manjul Bhargava – number theorist, master tabla player (classical Indian music)
- Noam Elkies – wicked smart Harvard mathematician, pianist, composer
- Half of the Holy Cross Math/CS Dept.
- Your Neighbor? Your Department? Your Kids?

Music is the pleasure the human soul experiences from counting without being aware that it is counting.
Gottfried Leibniz

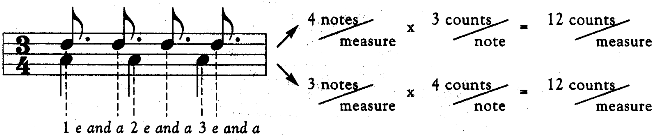
Counting a 2-against-3 rhythmic cycle



$$\mathrm{lcm}(2,3) = 6$$

Rhythm: Least Common Multiple

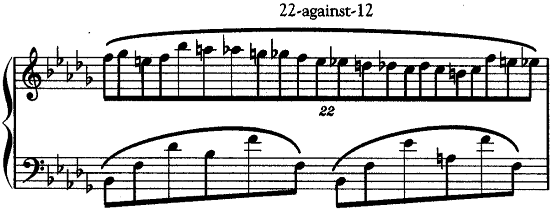
Counting a 3-against-4 rhythmic cycle

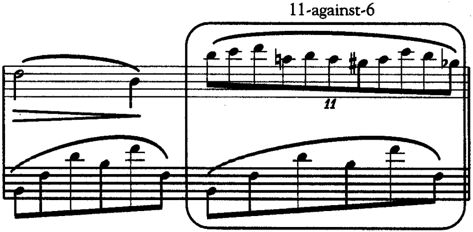


$$\mathrm{lcm}(3,4) = 12$$

Rhythm: Least Common Multiple

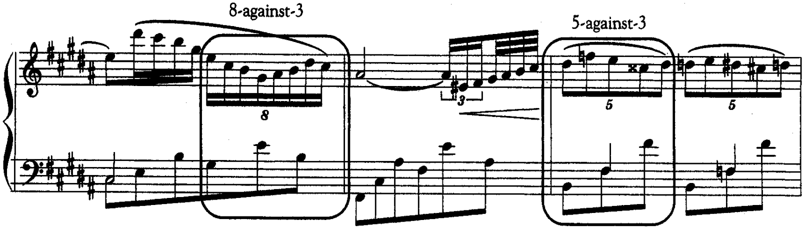
Music of Chopin





Rhythm: Least Common Multiple

Music of Chopin



$$\mathrm{lcm}(a, b) = \frac{ab}{\gcd(ab)}$$

Polyrhythmic Music

- Multiple rhythms at once
- African music often polyrhythmic with different drums and percussion instruments playing different rhythms simultaneously
- Classical Indian Music: tabla (pair of small hand drums). Drummers often play challenging combinations such as 11 beats in one hand and 12 in the other. Prime numbers play an important role.

Figure: A tabla

Figure: A triple dot in the Verdi Requiem.

Figure: A quadruple dot in the Verdi Requiem!

Rhythm: Geometric Series

Quadruple Dot $\Longleftrightarrow$ Geometric Series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16} \text{ beats}$$

$$a_0 + a_0 r + a_0 r^2 + \cdots + a_0 r^{n-1} = \frac{a_0(1 - r^n)}{1 - r}$$

What length does an infintely dotted quarter note get?

Answer: 2 beats or 1 half note (infinite geometric series)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - 1/2} = 2$$

Symmetry in Music: Group Theory

How to get more music out of a little motif:

Translations (shifting graph vertically) $\Longleftrightarrow$ Transpositions (shifting notes up or down)
Ex: Ballpark Music

Vertical Reflection (symmetry between right and left) $\Longleftrightarrow$ Retrograde (music same forward and backward)
Ex: Lean on Me

Horizontal Reflection (symmetry between top and bottom) $\Longleftrightarrow$ Inversion (what goes up, must come down)
Ex: Bach, Bach and more Bach

Symmetry in Music: Retrograde

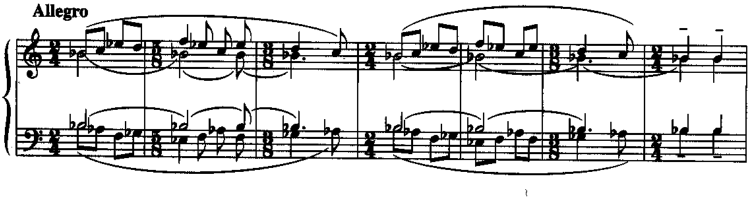
Figure: Joseph Haydn, *Piano Sonata No. 41, Hob. XVI/26*, Minuet

Symmetry in Music: Retrograde

Figure: Johann Sebastian Bach, *A Musical Offering*

Symmetry in Music

George F. Handel, *Messiah, Hallelujah chorus* (lose retrograde, form of tone painting)



Béla Bartók, *Mikrokosmos, No. 141, Subject and reflection* (inversion)

Symmetry in Music: Rotation

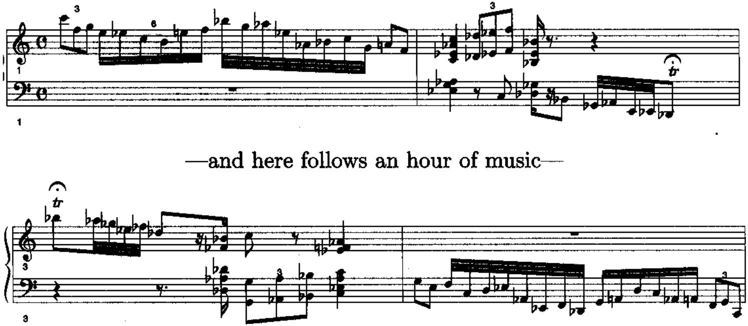


Figure: Paul Hindemith, *Ludas tonalis*, beginning and end.

Combining Symmetries

"I Got Rhythm" has an AABA structure, and a two-bar tag at the end.

We call these four equal sections A, A, B, and A. Three of the sections are the same, and one is different. Listen to the first section, and you'll be able

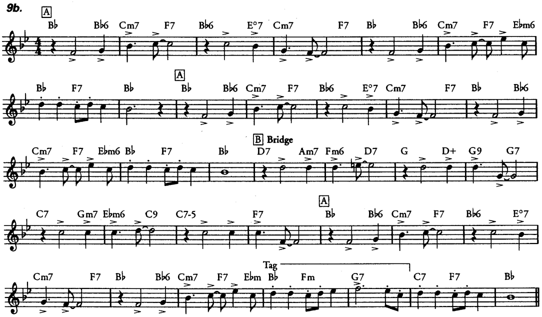


Figure: George Gershwin, *I Got Rhythm*, (transposition, retrograde and inversion, all in one song!)

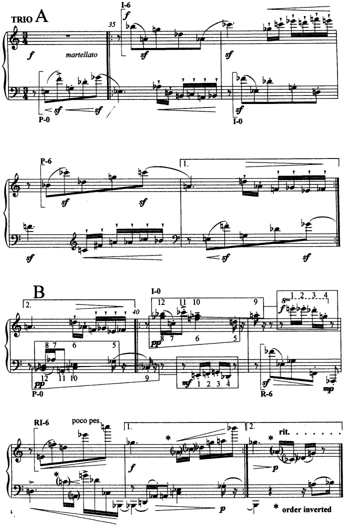


Figure: Arnold Schoenberg, *Piano Suite, Op. 25*, opening of the Minuet and Trio

Symmetry in Music: Twelve-tone Method

Figure: Tone Rows and their symmetries

Change Ringing (Bell Ringing)

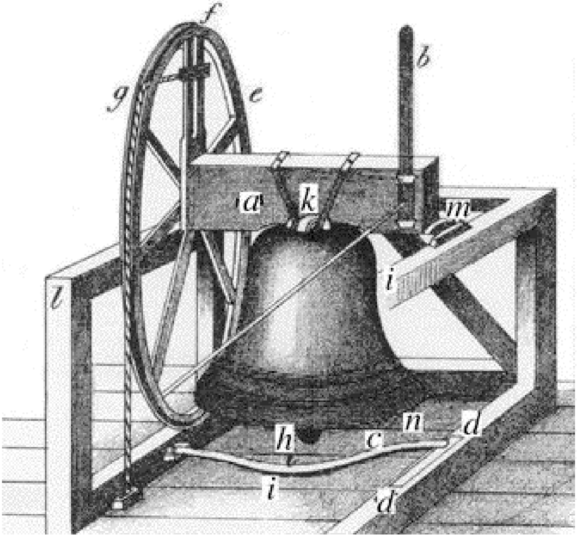


Figure: A typical church bell rung in the belfry.

Figure: Bell ringing practice in Stoke Gabriel parish church, south Devon, England.

Change ringing is a non-competitive and non-violent team activity that is highly stimulating intellectually and mildly demanding physically, and makes a beautiful sound. It develops mental and physical skills in a context of communal effort. The intense concentration required brings euphoric detachment that cleanses the mind of the day's petty demands and frustrations.
North American Guild of Change Ringers

Change Ringing: An Example

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
2 4 3 1	3 2 4 1	4 3 2 1
4 2 3 1	2 3 4 1	3 4 2 1
4 2 1 3	2 3 1 4	3 4 1 2
4 1 2 3	2 1 3 4	3 1 4 2
1 4 3 2	1 2 4 3	1 3 2 4
		1 2 3 4

Canterbury Minimus (true extent on 4 bells)
There are $4! = 24$ different possible rows. Each must be rung exactly once starting and ending with rounds (1 2 3 4).

Rules to ring an extent on n bells:

1. The first and last changes (rows) are rounds ($1\ 2\ 3\ 4 \cdots n$).
2. Other than rounds, all of the other $n!$ changes occur exactly once.
3. Between successive changes, no bell moves more than one position.
4. No bell rests for more than 2 (sometimes relaxed further to 4) positions.
5. Each working bell should do the same amount of "work" (obey the same overall pattern).
6. Horizontal symmetry should be present in the extent to help the ringers learn the path of their respective bell. This is called the *palindrome property*.

Note: Rules 1 - 3 are mandatory for an extent while Rules 4 - 6 are optional though often satisfied.

- A reordering of the numbers $1\ 2\ 3\ 4 \cdots n$ is called a permutation.
- How many possible changes on n bells?
  Answer: $n!$
- For n bells, how many "moves" are allowed?

$$n = 2 \qquad (12) \quad 1 \text{ move}$$

$$n = 3 \qquad (12), (23) \quad 2 \text{ moves}$$

$$n = 4 \qquad (12), (23), (34), (12)(34) \quad 4 \text{ moves}$$

$$n = 5 \quad (12), (23), (34), (45), (12)(34), (12)(45), (23)(45) \quad 7 \text{ moves!}$$

$$n = 6 \qquad (12), (23), (34), (45), (56)$$

$$(12)(34), (12)(45), (12)(56), (23)(45), (23)(56), (34)(56)$$

$$(12)(34)(56) \qquad \text{12 moves}$$

What's the pattern?

$$1, 2, 4, 7, 12 \ldots$$

Add one to our sequence:

$$2, 3, 5, 8, 13 \ldots$$

The Fibonnaci Sequence!

The number of allowable moves on n bells is $F_{n+1} - 1$.

n	$n!$	Approximate Duration	Name
3	6	15 secs.	*Singles*
4	24	1 mins.	*Minimus*
5	120	5 mins.	*Doubles*
6	720	30 mins.	*Minor*
7	5,040	3 hrs.	*Triples*
8	40,320	24 hrs.	*Major*
9	362,880	9 days	*Caters*
10	3,628,800	3 months	*Royal*
11	39,916,800	3 years	*Cinques*
12	479,001,600	36 years	*Maximus*

Table: Approximate duration to ring an extent on n bells and the names given to such an extent. Compositions: *Plain Bob Minimus*, *Grandshire Triples*

Change Ringing: 3 bells

The two extents on 3 bells:

1 2 3	1 2 3
2 1 3	1 3 2
2 3 1	3 1 2
3 2 1	3 2 1
3 1 2	2 3 1
1 3 2	2 1 3
1 2 3	1 2 3

Note the simple zig-zag pattern of Bell **1** in the first extent, sweeping easily from position 1 to position 3 and back again. We say that Bell **1** is *plain hunting.* It only needs to do this once to complete the extent. In this case, we say that the bell is "not working." Notice that in the second extent, Bell **1** follows a similar zig-zag path except that this begins on the second change.

Plain Bob Minimus (read down first, then hop to next column)

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3
		1 2 3 4

Let $a = (12)(34), b = (23), c = (34)$. The above sequence of 24 permutations can be "factored" as

$$[(ab)^3 ac]^3 \ = \ [abababa\, c]^3$$ Palindrome!