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Qualitative Theory of Dynamical Systems

Finiteness in the Planar Restricted Four-Body Problem

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In loving memory of Kate Zenda Santarelli

Abstract. Using BKK theory, we show that the number of equilibria (central configurations) in the planar, circular, restricted four-body problem is finite for any choice of masses. Moreover, the number of such points is bounded above by 196.

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1. Introduction

Certain finiteness questions in the field of celestial mechanics have been effectively tackled by employing techniques form Bernstein-Khovanskii-Kushnirenko (BKK) theory. Most notable is the work of Moeckel [11], introducing these concepts to the field while proving Saari's conjecture for the planar, three-body problem, as well as that of Hampton and Moeckel [5], showing finiteness for the number of relative equilibria (up to symmetry) in the four-body problem. In [16], it is shown that no solution to the planar, circular, restricted three-body problem can travel along a level curve of the amended potential function (Saari's conjecture modified to the restricted case). This is accomplished by applying a theorem of Bernstein's and performing all the required calculations by hand. Using similar methods, this

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result has recently been extended by Bruder [3] to the planar, circular, restricted four-body problem.

The finiteness questions referred to above can each be described as a complicated system of polynomial equations in a small (between 2 and 6) number of variables, whose coefficients depend on parameters (eg. the values of the masses). The typical goal is to show that the given system has a finite number of solutions for *all* choices of masses and to obtain an upper bound on the number of physically relevant solutions. BKK theory provides a relatively straight-forward approach for determining if a system of polynomial equations has a finite number of solutions for which all variables are nonzero. The techniques utilized involve computational questions in algebraic geometry, such as the computation of Newton polytopes. The distinct advantage of using these techniques, as opposed to applying Gröbner bases or resultants, is that they work for *generic* systems of polynomials. This allows for deeper conclusions and more profound results.

In this paper we show that the number of equilibria in the planar, circular, restricted four-body problem (PCR4BP) is finite for any choice of masses. Alternatively, the number of central configurations in the $3+1$-body problem is finite. Our intention is to demonstrate the usefulness of BKK theory in studying this slippery problem. For the general n-body problem, finiteness is often quite difficult to verify. For example, if a negative mass is allowed, a continuum of relative equilibria exists in the planar five-body problem [15].

Our main result differs from that of Hampton and Moeckel since $m_4 = 0$ in our problem, a case necessarily excluded in [5]. Although there has been a substantial amount of analytic and numerical work involving the equilibria of the PCR4BP [1, 7, 13, 18], our generic result on finiteness appears to be new. A nice summary of results on the problem is given in the introduction of Leandro's recent paper [7]. We also obtain an upper bound of 196 for the number of equilibria using BKK estimates. Most of these solutions are complex or physically meaningless (for example, a negative distance) as our numerical work indicates the true number is between 8 and 10 depending on the values of the masses. These lower estimates are confirmed by many researchers in the field, in particular by Pedersen [13] and Simó [18].

When necessary, symbolic and numerical calculations were performed using Maple [9] and Matlab [10], and confirmed utilizing the free mathematics software Sage [17]. Two Sage worksheets containing all the pertinent computations are available at `http://mathcs.holycross.edu/~groberts/Papers/papers.html`

2. The Planar, Circular, Restricted Four-body Problem

There are two possible circular, restricted four-body problems, depending upon which two of the three-body relative equilibria, Euler's collinear or Lagrange's equilateral triangle, is chosen. Palmore [12] studied the collinear case in general, showing that there are always $n + 3$ locations to continue a collinear relative equilibrium of the n-body problem into the full $n+1$-body problem. For $n=3$, four of these points lie on the line containing the collinear relative equilibrium while the other two are positioned symmetrically about this line. The more difficult problem arises when choosing the Lagrange equilateral triangle solution as the orbit for the three large masses [6]. We will refer to this problem as the planar, circular, restricted four-body problem (PCR4BP).

The PCR4BP consists of three large bodies (arbitrary mass) at the vertices of an equilateral triangle rotating on circular orbits about their common center of mass. A fourth infinitesimal mass, subject to the gravitational attraction of the three large "primaries", is inserted into their plane of motion and is assumed to have no effect on their circular orbits. Without loss of generality, we take both the total mass and the rotational frequency of the three large primaries to be one (ie. period 2π). Then, in order to be a solution of the three-body problem, the distance between each primary must also be one. The equations of motion for the fourth mass are taken in a rotating frame traveling at the same speed as the primaries and revolving about their center of mass. In this frame, the primaries are fixed at the positions $\mathbf{q}_1 = (\sqrt{3}/3, 0)$, $\mathbf{q}_2 = (-\sqrt{3}/6, 1/2)$ and $\mathbf{q}_3 = (-\sqrt{3}/6, -1/2)$. Their masses are m_1, m_2 and m_3 respectively, with the additional constraint that $m_1 + m_2 + m_3 = 1$.

Let (x, y) denote the position of the fourth body in the rotating frame. We introduce the variables a, b and c, representing the distances from the infinitesimal particle to the first, second and third primary, respectively (see Figure 1). These distance variables are not independent and are required to satisfy the important constraint

$$F = a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) + 1 = 0 \tag{1}$$

which can be derived using the Cayley-Menger determinant. Relation (1) ensures that the values of a, b, c describe a planar, rather than a spatial configuration. Given that the constraint is satisfied, the expressions

$$x = \frac{\sqrt{3}}{6}\left(b^2 + c^2 - 2a^2\right), \quad y = \frac{1}{2}\left(c^2 - b^2\right)$$

return the rectangular coordinates of the infinitesimal mass.

The equations of motion describing the trajectory of the infinitesimal mass in the rotating frame are given by

$$\begin{aligned} \ddot{x} &= 2\dot{y} + V_x \\ \ddot{y} &= -2\dot{x} + V_y \end{aligned}$$

where

$$V(x, y) = \frac{1}{2}\left((x - c_x)^2 + (y - c_y)^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c}$$

is the **amended potential** and (c_x, c_y) is the center of mass of the three fixed primaries. It is easy to check that the quantity $E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V$ is conserved for the above system. This is the equivalent of the Jacobi integral for the restricted

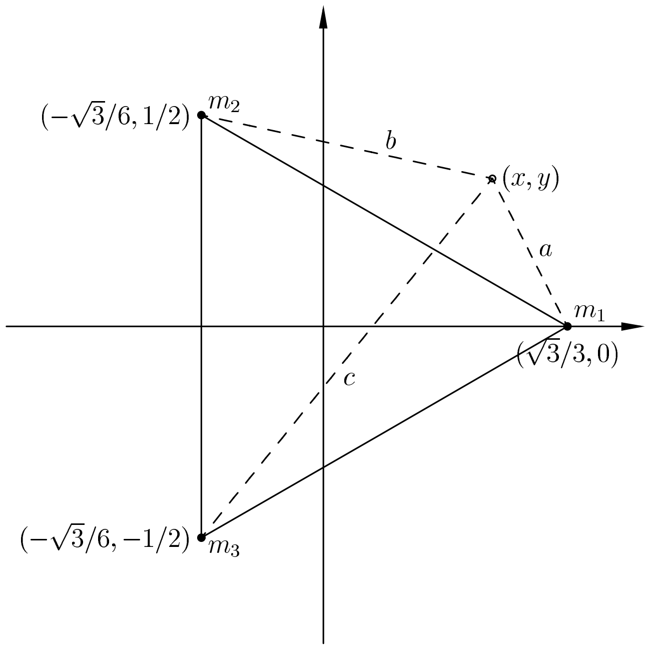

FIGURE 1. The set up for the PCR4BP.

three-body problem. In our distance variables, the amended potential function transforms nicely to

$$V = \frac{1}{2}\left(m_1 a^2 + m_2 b^2 + m_3 c^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c} + K$$

where K is the constant (using the relation $m_1 + m_2 + m_3 = 1$)

$$K \;=\; \frac{1}{2}(c_x^2 + c_y^2) - \frac{1}{6} \;=\; -\frac{1}{2}\left(m_1 m_2 + m_1 m_3 + m_2 m_3\right).$$

2.1. Equilibria

Any critical point of V is immediately an equilibrium point for the PCR4BP. These special **libration points** can be interpreted as ideal "parking spaces" for observational spacecraft. Each such point yields a **relative equilibrium** for the $3+1$-body problem, a configuration of four bodies rigidly rotating in the plane. More generally, given p nondegenerate critical points of V (the Hessian is invertible at each critical point), we can apply the Implicit Function Theorem to continue the entire configuration into the full $3+p$-body problem, generating a family of **central configurations** with three large masses near the vertices of an equilateral triangle and p small ones each close to an equilibrium point of the PCR4BP.

Our main result is the following theorem.

Theorem 2.1. *The number of equilibria in the planar, circular, restricted four-body problem is finite for any choice of masses. In particular, there are at most 196 equilibrium points.*

To find the critical points of V subject to the constraint $F = 0$, we solve the system of equations $\{\nabla V + \frac{1}{2}\lambda\nabla F = 0,\ F = 0\}$. This yields the system

$$\begin{aligned} m_1(1 - \frac{1}{a^3}) + \lambda(2a^2 - b^2 - c^2 - 1) &= 0 \qquad (2)\\ m_2(1 - \frac{1}{b^3}) + \lambda(2b^2 - a^2 - c^2 - 1) &= 0 \qquad (3)\\ m_3(1 - \frac{1}{c^3}) + \lambda(2c^2 - a^2 - b^2 - 1) &= 0 \qquad (4)\\ a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) + 1 &= 0. \end{aligned}$$

Clearing the denominators in the first three equations above yields a polynomial system of four equations in the four variables a, b, c and λ. We will refer to this polynomial system as system (I).

Summing equations (2), (3) and (4) yields an expression for λ given by

$$\lambda = \frac{1}{3}\left(1 - \frac{m_1}{a^3} - \frac{m_2}{b^3} - \frac{m_3}{c^3}\right). \qquad (5)$$

Substituting (5) into equations (2) and (3) and clearing denominators produces a system of three polynomial equations in the three distance variables a, b and c.

$$\begin{aligned} 2a^5b^3c^3 - 2m_3a^5b^3 - 2m_2a^5c^3 - a^3b^5c^3 + m_3a^3b^5 &- a^3b^3c^5\\ + (3m_1 - 1)a^3b^3c^3 + m_3a^3b^3c^2 + m_3a^3b^3 + m_2a^3b^2c^3 &+ m_2a^3c^5 \qquad (6)\\ + m_2a^3c^3 - 2m_1a^2b^3c^3 + m_1b^5c^3 + m_1b^3c^5 - 2m_1b^3c^3 &= 0 \end{aligned}$$

$$\begin{aligned} 2a^3b^5c^3 - 2m_3a^3b^5 - 2m_1b^5c^3 - a^5b^3c^3 + m_3a^5b^3 &- a^3b^3c^5\\ + (3m_2 - 1)a^3b^3c^3 + m_3a^3b^3c^2 + m_3a^3b^3 + m_1a^2b^3c^3 &+ m_1b^3c^5 \qquad (7)\\ + m_1b^3c^3 - 2m_2a^3b^2c^3 + m_2a^5c^3 + m_2a^3c^5 - 2m_2a^3c^3 &= 0 \end{aligned}$$

$$a^4 + b^4 + c^4 - (a^2b^2 + a^2c^2 + b^2c^2) - (a^2 + b^2 + c^2) + 1 = 0. \qquad (8)$$

We will refer to the system of equations (6), (7) and (8) as system (II). The first two equations have been written in order to display an important symmetry whereby interchanging a and b as well as m_1 and m_2 in equation (6) yields equation (7) and vice versa.

Since our variables correspond to distances from the primaries, we can ignore solutions to the above systems for which one or more of a, b, c vanish. We will refer to any solution with at least one variable equal to zero as **trivial**. Denote $\mathbb{C}^* = \mathbb{C} - \{0\}$. It is straight-forward to show that systems (I) and (II) are equivalent. Specifically, $(a, b, c) \in (\mathbb{C}^*)^3$ is a solution to system (II) if and only if $(a, b, c, \lambda) \in (\mathbb{C}^*)^4$ is a solution to system (I), where λ is given by equation (5). The only subtlety to the argument lies in showing that $\lambda \neq 0$. This is taken care of by the following lemma.

Lemma 2.2. *There are no solutions (real or complex) to system (I) with $\lambda = 0$.*

Proof. Substituting $\lambda = 0$ into equations (2), (3) and (4) implies that a, b and c are each cube roots of unity, $\zeta_j = e^{i(2\pi j/3)}$. Using Gröbner bases, one can check that the polynomial ideal $I =< a^3-1, b^3-1, c^3-1, F >$ in $\mathbb{C}[a,b,c]$ is actually the entire polynomial ring. This implies that the variety is empty and thus no such solution to system (I) can exist with $\lambda = 0$.

Alternatively, a more direct approach is to show that F does not vanish on each of the 27 possible three-tuples of ζ_j's. The fact that F is a symmetric polynomial simplifies the computations greatly. If a, b and c are distinct cube roots of unity, then $F = 1 \neq 0$ follows quickly. If at least two of the variables are equal (without loss of generality take $a = b$), then F reduces to

$$a^4 + c^4 - 2a^2c^2 - 2a^2 - c^2 + 1 \; = \; (a^2 - c^2 - 1)^2 - 3c^2. \tag{9}$$

By substituting $c = \zeta_j$ into (9) for each j, we obtain three simple polynomials in a, none of which vanishes at a cube root of unity. □

2.2. Equal Masses

As a relatively simple example, we consider the case $m_1 = m_2 = m_3 = 1/3$ and show that there are precisely 10 equilibria, all lying on an axis of symmetry. According to Leandro, this was first proven by Lindow [8] in 1922. The result also appears, without proof, in Arenstorf [1].

We begin by showing that all real, positive solutions to system (I) lie on a line of symmetry, that is, either $a = b$, $a = c$ or $b = c$. Consider the polynomial $g = (a-b)(a-c)(b-c) \cdot g_1$ where

$$g_1 = c^2(a^2 + ab + b^2)(a + b + c) + ab(a + b)(ab + ac + bc).$$

Using Gröbner bases, one can check that g is contained in the ideal generated by the four polynomials of system (I) when $m_1 = m_2 = m_3 = 1/3$. If an element in the variety of this ideal did not possess any symmetry, then it would necessarily have to be a zero of g_1. However, g_1 can never vanish if a, b, c are each real and positive. It follows that $a = b$, $a = c$ or $b = c$ for any physically relevant solution. The polynomial g was found by simplifying the difference of equations (2) and (3) as well as (2) and (4) and then eliminating λ from the resulting expressions.

In the equal mass case, the four polynomials of system (I) are invariant under any permutation of (a, b, c). Thus, solutions come in groups of three, (η, η, ζ), (η, ζ, η) or (η, ζ, ζ). Geometrically, this group of solutions is invariant under a 120° rotation, with each element lying on a different altitude of the equilateral triangle formed by the primaries. The only exception is the solution $a = b = c = 1/\sqrt{3}$ which gives a critical point at the origin, the center of the triangle.

To find the remaining positive, real solutions, we substitute $b = a$ and $m_1 = m_2 = m_3 = 1/3$ into system (II). Equations (6) and (7) become identical, and the term $a^3(a - c)$ factors out of the polynomial in equation (6). Since we have already

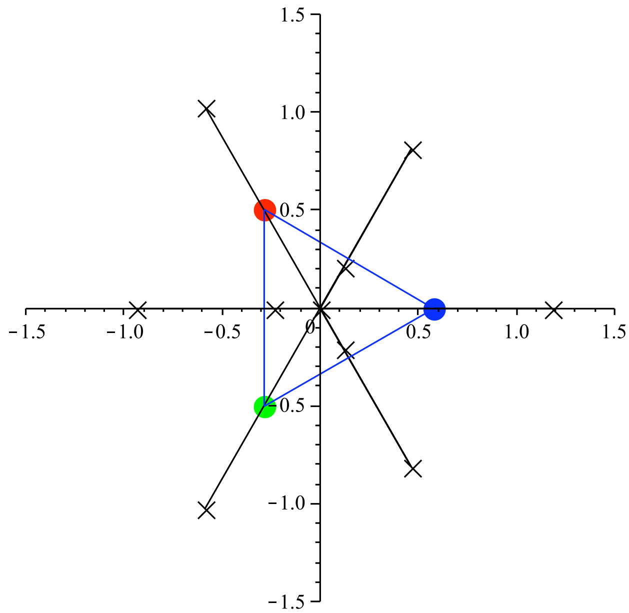

FIGURE 2. The 10 symmetric equilibria (indicated by an $\times$) for the equal mass case.

accounted for the only solution with all three variables equal, we can cancel this factor from the first equation. The resulting system is

$$\begin{aligned} 3a^4c^3 - a^4 + 3a^3c^4 - a^3c + a^2 - 2ac^3 + ac - 2c^4 + c^2 &= 0 \\ a^4 - 2a^2c^2 - 2a^2 + c^4 - c^2 + 1 &= 0. \end{aligned}$$

Computing a Gröbner basis for this system with a lex order ($a > c$) produces a polynomial in c that has 22 nonzero roots. Of these roots, five are real and positive, but only three correspond to positive a-values. The three physically relevant solutions (a, c) are

$$\{(0.502465683134481, 0.816308578384975),\ (1.55145156726892, 0.602648135699807),$$
$$(0.817298143882299, 1.51253623586187)\}.$$

The first point lies inside the triangle formed by the primaries. Taken with the three-fold symmetry and the solution at the origin, this yields a precise total of 10 equilibria for the equal mass case (see Figure 2). These values were computed with Maple and confirmed using Sage. For details, see the Sage worksheet "Equal Mass case" available at `http://mathcs.holycross.edu/~groberts/Papers/papers.html`

3. BKK Theory

To show that the number of solutions to system (I) or (II) is finite for all masses, we utilize BKK theory, as demonstrated so effectively by Hampton and Moeckel in [5]. We regard an element $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$ as an exponent vector of the monomial $z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$, abbreviated simply as z^k. A polynomial $f \in \mathbb{C}[z_1, \ldots z_n]$ is a sum of monomials, generically written as

$$f = \sum_k c_k z^k$$

where $c_k \in \mathbb{C}$ for each k and there are only a finite number of terms in the sum. The Newton polytope for f, denoted $N(f)$, is the convex hull in $\mathbb{R}^n$ of the set of all exponent vectors k occurring for f.

3.1. Bernstein's Theorem and Puiseux Series

Suppose that $r = (r_1, r_2, \ldots, r_n)$ is a solution to the system of m polynomial equations

$$\begin{aligned} f_1(z_1, \ldots, z_n) &= 0 \\ f_2(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_m(z_1, \ldots, z_n) &= 0, \end{aligned} \tag{10}$$

that is, r belongs to the affine variety $\mathbb{V}(f_1, \ldots, f_m)$. We say that r lies in the **algebraic torus** $\mathbb{T}$ if $r_i \in \mathbb{C}^* \ \forall i$. One of the big advantages to using distance variables is that we are only concerned with solutions in $\mathbb{T}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector of rational numbers. For a given polynomial f, the **reduced polynomial** f_α is the sum of all terms of f whose exponent vectors k satisfy

$$\alpha \cdot k = \min_{l \in N(f)} \alpha \cdot l.$$

This equation defines a face of the polytope $N(f)$ with inward pointing normal α, although this face is not necessarily of codimension one. For all exponent vectors k on this face, the dot product $\alpha \cdot k$ will be strictly smaller than the dot product of α with any exponent vector elsewhere in $N(f)$. For example, for the vectors $\alpha = (1, 1, 0)$, $\beta = (2, 1, 0)$ and $\gamma = (1, 1, 1)$, the reduced polynomials for the constraint F are

$$F_\alpha = F_\beta = c^4 - c^2 + 1 \quad \text{and } F_\gamma = 1.$$

The reduced polynomials are considerably easier to handle than the original ones. Note also that different choices of α can induce the same polynomial.

For a given rational vector $\alpha = (\alpha_1, \dots, \alpha_n)$, the **reduced equations** for system (10) are defined using the reduced polynomials corresponding to α:

$$\begin{aligned} f_{1\alpha}(z_1, \dots, z_n) &= 0 \\ f_{2\alpha}(z_1, \dots, z_n) &= 0 \\ &\vdots \\ f_{m\alpha}(z_1, \dots, z_n) &= 0. \end{aligned} \tag{11}$$

Bernstein makes use of these reduced equations in the following theorem [2]. A readable proof using algebraic geometry and Puiseux series can be found in [11].

Theorem 3.1. *Suppose that system* (10) *has infinitely many solutions in* $\mathbb{T}$*. Then there exists a vector* $\alpha = (\alpha_1, \dots, \alpha_n)$ *with* $\alpha_i \in \mathbb{Q}$ *and* $\alpha_j = 1$ *for some* j*, such that the system of reduced equations* (11) *also has a solution in* $\mathbb{T}$*.*

To apply this theorem successfully, we must rule out all possible rational vectors α. Fortunately, although there are an infinite number of choices for α, there are only a finite number of reduced systems to check. This is due to the fact that different vectors give rise to the same system of reduced equations. To ensure that all possible systems are considered, we first compute the normal fan of the Minkowski sum polytope

$$MS = P_1 + P_2 + \cdots + P_m = \{q \in \mathbb{R}^n : q = q_1 + q_2 + \cdots + q_m, q_i \in P_i\}$$

where $P_i = N(f_i)$ is the Newton polytope for f_i. It then suffices to check all inward normals α of facets of MS as well as those vectors corresponding to lower dimensional faces (eg. edges, vertices, etc.) See [5] and [11] for details.

While Theorem 3.1 is often sufficient to prove finiteness (as is the case with the problems considered in [3], [11] and [16]), it could be the case that some vector α yields a reduced system that actually has a solution in $\mathbb{T}$, that is, a nontrivial solution. Such a difficulty occurs in [5] as well as for our problem. In this case, we introduce complex Puiseux series in order to rule out these problematic vectors. Such a series is defined as

$$z(t) = \sum_{i=i_0}^{\infty} a_i t^{\frac{i}{q}}$$

where $a_i \in \mathbb{C}$, and $q \in \mathbb{N}$, $i_0 \in \mathbb{Z}$ are fixed. For example,

$$z_1(t) = t \tag{12}$$

$$z_2(t) = -2 + t^{1/3} - 5i\, t^{2/3} + \cdots \tag{13}$$

$$z_3(t) = 5t^{-1/2} - 4 + 3t^{1/2} + \cdots \tag{14}$$

are each complex Puiseux series. The key fact, proved in [5], is that if system (10) has an infinite variety in $\mathbb{T}$, then it contains a Puiseux series solution $z_j(t), j = 1, \dots n$, convergent in some punctured neighborhood of $t = 0$. Moreover, at least one of the variables is simply $z_j(t) = t$. The **order** of this solution is the vector of rationals α formed by the fractional exponent of the first term in each series. For the three series $z_1(t), z_2(t), z_3(t)$ described in equations (12), (13) and (14), the order is the vector $\alpha = (1, 0, -1/2)$. For the case when a Puiseux series solution exists, the vector α generates a system of reduced equations (11) that will have a nontrivial solution given by the coefficients a_{i_0} of the first term in each series.

3.2. Proof of Theorem 2.1

Proof. To prove Theorem 2.1, we first show that the number of solutions to system (II) is finite. Since our two systems are equivalent, this will prove finiteness for the number of equilibria in the PCR4BP. We will show that for all choices of vectors α with at least one component positive, either the reduced equations have only trivial solutions (at least one component zero) or if a nontrivial solution does exist, then there is no Puiseux series solution to the original system with order α. Fortunately, as described above, we only need to consider those vectors representing a facet (as an inward normal) or lower dimensional face (edges or vertices) of the Minkowski sum polytope MS. Remarkably, the only assumption on the masses required for finiteness is $m_j \neq 0$. In other words, Theorem 2.1 holds for any set of masses in $\mathbb{C}^*$ provided $m_1 + m_2 + m_3 = 1$.

We compute the Minkowski sum polytope MS for the three polynomials in system (II) using Matlab, which calls the software Qhull [14]. Calculations were also confirmed with Sage. The three-dimensional polytope MS has 12 vertices, 24 edges and 14 facets (see Figure 3). Due to symmetry, the inward pointing normals come in pairs $\pm\alpha$. By Theorem 3.1, we can exclude any inward normals with all components nonpositive. This results in the following list of inward normals:

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (1, 1, 0),\ (1, 0, 1),\ (0, 1, 1),\ (1, 1, 1).$$

Since system (II) is invariant under the symmetry transformation $a \leftrightarrow b$, $m_1 \leftrightarrow m_2$, we can exclude the vectors $(0, 1, 0)$ and $(1, 0, 1)$ by studying their symmetric counterparts $(1, 0, 0)$ and $(0, 1, 1)$, respectively. This symmetry also reduces the number of edges that have to be considered. Of the five inward normals left to be studied, two of them, $(1, 0, 0)$ and $(0, 0, 1)$, have reduced equations with nontrivial solutions. The others give rise to reduced systems that have no solutions in $\mathbb{T}$.

For example, the reduced equations (factored) for the inward normal $(0, 1, 1)$ are

$$\begin{aligned} a^4 - a^2 + 1 &= 0 \\ -a^3(2a^2 - 1)(m_3 b^3 + m_2 c^3) &= 0 \qquad (15) \\ a^3(m_3 a^2 b^3 + m_2 a^2 c^3 + m_3 b^3 - 2m_2 c^3) &= 0. \qquad (16) \end{aligned}$$

The polynomials $a^4 - a^2 + 1$ and $2a^2 - 1$ have no common roots. By equation (15), this implies that $m_3 b^3 = -m_2 c^3$. Substituting this relation into equation (16) gives $-3m_2 a^3 c^3 = 0$ which implies $c = 0$ and therefore $b = 0$. Alternatively, computing a Gröbner basis with a standard lexicographic order gives $\{m_2 c^3, m_3 b^3, a^4 - a^2 + 1\}$ which quickly yields $b = c = 0$. Either argument serves to eliminate the inward normal vector $(0, 1, 1)$. A similar calculation eliminates the vector $(1, 1, 0)$. The inward normal vector $\alpha = (1, 1, 1)$ yields an inconsistent reduced system $F_\alpha =$

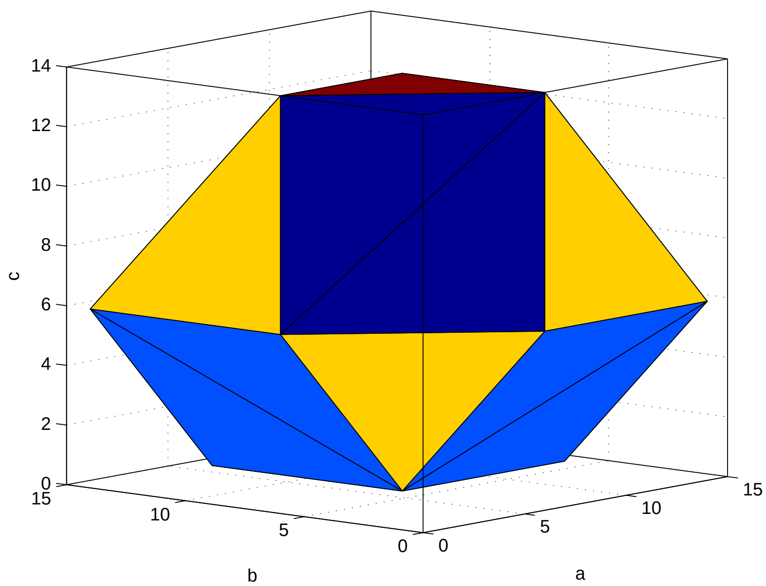

FIGURE 3. The Minkowski sum polytope MS for the three polynomials of system (II).

$1 = 0$. In addition to eliminating $(1,1,1)$ from contention, this also serves to eliminate any edge contained in this facet. As Moeckel points out in [11], for any vector α corresponding to a lower dimensional face contained in a trivial facet (one for which some reduced equation contains only one term), the reduced equations corresponding to α will also be trivial. In this case, any edge or vertex lying on the facet with inward normal $(1,1,1)$ yields the inconsistent reduced equation $F_\alpha = 1 = 0$.

To rule out the remaining inward normals $(1,0,0)$ and $(0,0,1)$, we must show that no Puiseux series solution to system (II) exists having these orders. We give the argument for $(1,0,0)$. The calculations excluding the vector $(0,0,1)$ are essentially identical.

The reduced equations for the inward normal vector $(1,0,0)$ contain all monomials missing the variable a. In factored form they are

$$\begin{aligned}
b^4 - b^2c^2 - b^2 + c^4 - c^2 + 1 &= 0 \\
m_1 b^3 c^3 (b^2 + c^2 - 2) &= 0 \\
m_1 b^3 c^3 (-2b^2 + c^2 + 1) &= 0
\end{aligned}$$

Given that $m_1 \neq 0$, the only solutions for these equations in $\mathbb{T}$ are of the form $(a, \pm 1, \pm 1)$ for any $a \in \mathbb{C}^*$. Thus, if there were an infinite number of solutions to system (II) in $\mathbb{T}$ with order $(1,0,0)$, then there would be a Puiseux series solution of the form $a = t, b = \pm 1 + \cdots, c = \pm 1 + \cdots$.

If we make the substitution $(a,b,c) = (t, z_2, z_3)$ into system (II), we obtain a system of three polynomial equations in two variables $z = (z_2, z_3)$ with coefficients that are polynomials in t. This system can be written compactly as

$$G(t,z) = G_0(z) + G_2(z)t^2 + \cdots + G_5(z)t^5 = 0$$

(there are no terms having just t as a coefficient.) The system $G_0(z)$ is precisely the reduced equations corresponding to $\alpha = (1,0,0)$. A hand computation (checked using Maple as well as Sage) shows that the Jacobian matrix $DG_0(\pm 1, \pm 1)$ has rank two (full rank). By the Implicit Function Theorem, there exists a C^∞ function $h : B \mapsto \mathbb{C}^2$ and a neighborhood B of 0, such that $G(t, h(t)) = 0 \; \forall t \in B$. Since h has derivatives of all orders, it follows that our Puiseux series solution is actually a power series. Substituting $z_2 = \pm 1 + ut^2 + \cdots$ and $z_3 = \pm 1 + vt^2 + \cdots$ into $G = 0$ yields an inconsistent system as the constraint equation becomes $-3t^2 + \cdots = 0$. This shows that no Puiseux series solution of order $\alpha = (1,0,0)$ exists.

We must also check the lower dimensional faces to rule out all possible vectors α. Only edges of the Minkoswki sum polytope MS need be considered since each vertex will yield reduced equations containing just a single monomial. Since the coefficients of all monomials obtained only vanish if $m_j = 0$, only trivial solutions are possible for the reduced equations corresponding to vertices of MS. This holds even though the choices $m_1 = 1/3$ and $m_2 = 1/3$ cause the monomial $a^3b^3c^3$ in equations (6) and (7) respectively, to vanish. This case never materializes however, because the exponent vector $(3,3,3)$ lies strictly inside the Newton polytope of each polynomial.

To find representative vectors for the edges of MS, we sum the inward normals on adjoining facets. Excluding those edges that have no representative vectors with at least one positive component reduces the list of 24 down to 15. Three of these remaining edges are part of the facet with inward normal $(1,1,1)$ and by the argument above, lead to an inconsistent set of reduced equations. Of the remaining 12 edges, 5 can be eliminated due to the symmetry between the first two variables. This leaves the following 7 edges, described by the sum of the inward normals of the two adjoining facets:

$$\begin{aligned}
(2,1,0) &= (1,0,0) + (1,1,0), & (1,-1,-1) &= (1,0,0) + (0,-1,-1)\\
(2,0,1) &= (1,0,0) + (1,0,1), & (-1,-1,1) &= (0,0,1) + (-1,-1,0)\\
(1,0,2) &= (0,0,1) + (1,0,1), & (1,-1,1) &= (1,0,1) + (0,-1,0)\\
& & (1,1,-1) &= (1,1,0) + (0,0,-1).
\end{aligned}$$

The reduced equations corresponding to each of the above vectors lead to only trivial solutions. For example, taking $\alpha = (2, 1, 0)$ gives the reduced equations

$$\begin{aligned} c^4 - c^2 + 1 &= 0 \\ m_1 b^3 c^3 (c^2 - 2) &= 0 \\ m_1 b^3 c^3 (c^2 + 1) &= 0 \end{aligned}$$

which requires $b = 0$. Similarly, the vector $\alpha = (1, -1, -1)$ yields the reduced system

$$\begin{aligned} b^4 - b^2 c^2 + c^4 &= 0 \\ m_1 b^3 c^3 (b^2 + c^2) &= 0 \\ -m_1 b^3 c^3 (2b^2 - c^2) &= 0 \end{aligned}$$

which necessitates $b = c = 0$. As a final example, for the vector $\alpha = (1, 1, -1)$, the constraint equation (8) is simply $c^4 = 0$ and thus only trivial solutions occur. The calculations to eliminate the remaining vectors are quite similar. This concludes the proof of finiteness.

To obtain an upper bound on the number of possible equilibria, we return to system (I) and compute the mixed volume of the given four polynomials in four variables. A mixed volume of 196 was computed with both Qhull and Sage. This value gives an upper bound on the number of solutions in $\mathbb{T}$ (real or complex) to system (I) and for generic coefficients, it is precisely the total number of solutions [4]. We have not missed any possible real solutions with a, b, c positive and $\lambda = 0$ due to Lemma 2.2. This completes the proof of Theorem 2.1. □

Remark 3.2. *The mixed volume for system (II) is 268, giving a slightly higher upper bound than 196. The reason for the larger number is due to the fact that there are relations between the coefficients of the polynomials in system (II). Cancellations can occur in the corresponding ideal that will lead to a system with smaller mixed volume. For example, one such system was found with a mixed volume of 220. However, applying BKK theory to prove finiteness was far easier for system (II) than system (I).*

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