LINEAR STABILITY IN THE $1 + N$-GON RELATIVE EQUILIBRIUM

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We study the linear stability of the relative equilibrium in the n-body problem consisting of n equal masses at the vertices of a regular n-gon with an additional body of mass m at the center. This configuration is shown to be linearly unstable when $n \leq 6$. For $n \geq 7$, a value h_n is found such that the configuration is linearly stable if and only if $m > h_n$. This value is shown to increase proportionately to n^3.

1 Introduction

A *relative equilibrium* is a special solution of the n-body problem which rotates rigidly about its center of mass if given the correct initial momentum. In rotating coordinates these special solutions become fixed points, hence the name relative equilibria. For example, when $n = 3$ the only noncollinear three-body relative equilibrium consists of each body, regardless of its mass, located at the vertex of an equilateral triangle. This configuration is linearly stable only when one of the masses dominates the other two.[2,10] This situation occurs in our galaxy with Jupiter, the sun, and the Trojan asteroids, where the sun is the dominant mass.

Moeckel has conjectured that a relative equilibrium can be linearly stable only if it contains a mass significantly larger than the others.[8] This is the case for the equilateral triangle. It is easy to see that the regular n-gon with a central mass is a relative equilibrium for any value m of the central mass (including $m = 0$) and thus it is natural to treat m as a parameter. Maxwell carried out such an analysis in his study of Saturn's rings and concluded that for sufficiently large values of m, the ring is linearly stable.[3,4] Moeckel provides a thorough analysis of the stability problem in general, making use of the special properties of the linearized matrix to obtain a factorization of the characteristic polynomial.[7] He then proceeds to use his techniques to analyze some specific, highly symmetric relative equilibria including the regular n-gon with a central mass, hereafter referred to as the $1 + n$-gon. (As a correction to Maxwell's study, Moeckel reveals that the $1 + n$-gon is linearly stable for sufficiently large m only when $n \geq 7$.)

We carry the analysis a step further. We investigate the cases $n = 3, 4, 5, 6$ to determine if they are linearly stable for *any* value of m, and for $n \geq 7$, we ask how large the central mass m must be relative to the mass of the bodies in the n-gon in order for the configuration to be linearly stable. It turns out that for the cases $n = 3, 4, 5, 6$, the $1 + n$-gon is not linearly stable for any values of m. For $n \geq 7$, the configuration is linearly stable if and only if m is greater than a certain value h_n, and as n increases, h_n increases like τn^3, where $\tau \approx 0.435$. In other words, the $1 + n$-gon relative equilibrium undergoes a bifurcation in stability at a value of m proportional to n^3.

For a simple application to our results, consider the nearly circular rotation of the moon about the Earth. The Earth's mass is approximately 81 times that of the moon's. Suppose that the moon was suddenly split into n equal pieces, with each "new" moon landing close to a vertex of a regular n-gon with the Earth at its center. Then the Earth would be $81n$ times heavier than any one of its moons. The question then becomes is the Earth heavy enough to maintain this configuration? The answer is no, unless $n \in \{7, 8, \ldots, 13\}$, as τn^3 grows faster than $81n$.

We would like to point out that similar results were obtained using different methods by Elmabsout.[1] At the time this paper was being completed, we were unaware of his work. The results here are more precise and presented in more detail.

2 Determining Linear Stability of Relative Equilibria

2.1 Relative equilibria in the n-body problem

We let the mass and position of the n bodies be given by m_i and $\mathbf{q}_i \in \mathbb{R}^2$, $i = 1, \ldots, n$, respectively. Let $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ be the distance between the ith and jth bodies and let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in \mathbb{R}^{2n}$. Using Newton's law of motion and the inverse square law for attraction due to gravity, the second-order equation for the ith body is given by

$$m_i \ddot{\mathbf{q}}_i = \sum_{i \neq j} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} = \frac{\partial U}{\partial \mathbf{q}_i},$$

where $U(\mathbf{q})$ is the Newtonian potential function:

$$U(\mathbf{q}) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}.$$

We let the momentum of each body be $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$ and let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{2n}$. The equations of motion can then be written as

$$
\begin{aligned}
\dot{\mathbf{q}} &= M^{-1}\mathbf{p} = \frac{\partial H}{\partial \mathbf{p}} \\
\dot{\mathbf{p}} &= \nabla U(\mathbf{q}) = -\frac{\partial H}{\partial \mathbf{q}}
\end{aligned}
\tag{1}
$$

where M is the diagonal mass matrix with diagonal $m_1, m_1, m_2, m_2, \ldots, m_n, m_n$ and $H(\mathbf{q}, \mathbf{p})$ is the Hamiltonian function:

$$
H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{n} \frac{\|\mathbf{p}_i\|^2}{2m_i} - U(\mathbf{q}) = \frac{1}{2}\mathbf{p}^{\mathrm{T}} M^{-1}\mathbf{p} - U(\mathbf{q}).
$$

Next, let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ so that $e^{\omega J t} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$. To introduce coordinates that uniformly rotate with period $2\pi/\omega$, we let $\mathbf{x}_i = e^{\omega J t}\mathbf{q}_i$ and $\mathbf{y}_i = e^{\omega J t}\mathbf{p}_i$. This is a symplectic change of variables preserving the Hamiltonian structure of system (1).[5] The new system becomes

$$
\begin{aligned}
\dot{\mathbf{x}} &= \omega K \mathbf{x} + M^{-1}\mathbf{y} = \frac{\partial \hat{H}}{\partial \mathbf{y}} \\
\dot{\mathbf{y}} &= \nabla U(\mathbf{x}) + \omega K \mathbf{y} = -\frac{\partial \hat{H}}{\partial \mathbf{x}}
\end{aligned}
\tag{2}
$$

where K is a $2n \times 2n$ block diagonal matrix with J on the diagonal and $\hat{H}(\mathbf{x}, \mathbf{y})$ is the Hamiltonian function:

$$
\hat{H}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{T}} M^{-1}\mathbf{y} - U(\mathbf{x}) - \omega\, \mathbf{x}^{\mathrm{T}} K \mathbf{y}.
$$

While system (1) has no equilibria, system (2) does. An equilibrium point in this new system will correspond to a periodic solution in the n-body problem consisting of a configuration of masses which rotates rigidly about its center of mass. Using the fact that $KM = MK$ and $K^2 = -I$, an equilibrium $(\mathbf{x}, \mathbf{y})$ of system (2) must satisfy $\mathbf{y} = -\omega M K \mathbf{x}$ and

$$
\nabla U(\mathbf{x}) + \omega^2 M \mathbf{x} = 0 \tag{3}
$$

A *relative equilibrium* of the n-body problem is a configuration $\mathbf{x} \in \mathbb{R}^{2n}$ which satisfies the algebraic equations in (3) for some value of ω.

2.2 Linear stability

Linearizing system (2) about a relative equilibrium $(\mathbf{x}, \mathbf{y})$ yields the matrix

$$A = \begin{bmatrix} \omega K & M^{-1} \\ S & \omega K \end{bmatrix} \tag{4}$$

where $S = D\nabla U(\mathbf{x})$ is a $2n \times 2n$ symmetric matrix. The characteristic polynomial of A, $P(\lambda)$, is of degree $4n$ and is an even polynomial, since A is a Hamiltonian matrix.[5] Suppose that $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, and write $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^{2n}$. The eigenvector equation $A\mathbf{v} = \lambda \mathbf{v}$ then reduces to

$$\begin{aligned} \mathbf{v}_2 &= M(\lambda I - \omega K)\mathbf{v}_1 \\ B\mathbf{v}_1 &= 0 \end{aligned}$$

where

$$B = M^{-1}S + (\omega^2 - \lambda^2)I + 2\lambda\omega K. \tag{5}$$

Therefore, to obtain the eigenvalues of A, one need only take the determinant of B and find the roots. In other words, $P(\lambda) = \mathrm{Det}(B)$.

We will call two vectors $\mathbf{v}$ and $\mathbf{w}$ M-orthogonal if $\mathbf{v}^{\mathrm{T}} M \mathbf{w} = 0$. Direct calculation shows that $M^{-1}S$ and K are symmetric and skew-symmetric, respectively, with respect to an M-orthonormal basis. Writing B with respect to an M-orthonormal basis and taking the transpose will not change its determinant. This gives an alternative argument for showing $P(\lambda) = P(-\lambda)$. Throughout this work, $P(\lambda)$ will denote the characteristic polynomial for A:

$$P(\lambda) = \det\left[M^{-1}S + (\omega^2 - \lambda^2)I + 2\lambda\omega K\right]. \tag{6}$$

Moeckel's idea is to obtain a factorization for $P(\lambda)$ by finding a subspace of $\mathbb{R}^{2n}$ which is invariant for both $M^{-1}S$ and K.[7] Suppose that W is a subspace of $\mathbb{R}^{2n}$ such that $M^{-1}SW = KW = W$. Letting $W^{\perp} = \{\mathbf{v} \in \mathbb{R}^{2n} : \mathbf{v}^{\mathrm{T}} M \mathbf{w} = 0 \ \forall \mathbf{w} \in W\}$ be the orthogonal complement of W with respect to M, direct calculation shows that $M^{-1}SW^{\perp} = KW^{\perp} = W^{\perp}$. If T_1 and T_2 are M-orthonormal basis for W and $W^{\perp}$ respectively, then the matrix B written with respect to the basis $T = T_1 \cup T_2$ is

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where B_1 and B_2 are the restrictions of B to the subspaces W and $W^{\perp}$ with respect to the basis T_1 and T_2, respectively. It follows that

$$\det(B) = \det(B_1) \cdot \det(B_2)$$

or $P(\lambda) = P_1(\lambda)P_2(\lambda)$. Moreover, since writing $M^{-1}S$ and K with respect to an M-orthonormal basis yields a symmetric and a skew-symmetric matrix respectively, the same argument used above to show P was even applies here to P_1 and P_2.

Proposition 2.1 *Suppose that $W \subset \mathbb{R}^{2n}$ is an invariant subspace for both $M^{-1}S$ and K. Then the stability polynomial can be factored into two even polynomials in λ, $P(\lambda) = P_1(\lambda)P_2(\lambda)$, each given by equation (6) with the matrices involved restricted to the subspaces W and $W^{\perp}$, respectively.*

This proposition is not very useful if we can't "guess" invariant subspaces for both $M^{-1}S$ and K simultaneously. Moeckel successfully does this with the collinear, regular polygon, and $1+n$-gon relative equilibria.[7] Building on the work of Palmore[9], he utilizes the special structure of the matrix S in these cases to find two and four-dimensional invariant subspaces. He finds enough subspaces to calculate all of the eigenvalues and then performs a thorough analysis to classify their stability. In the next section, we will describe these subspaces for the $1+n$-gon and obtain a factorization of $P(\lambda)$. In Section 4, we locate the precise value for which the $1+n$-gon becomes linearly stable. In Section 5, we provide the estimates on this bifurcation value to show it is asymptotic to τn^3.

3 The $1+n$-Gon

3.1 Invariant subspaces and factoring $P(\lambda)$

The $1+n$-gon is the relative equilibrium consisting of n equal masses at the vertices of a regular n-gon with an additional mass at the center. We set $m_k = 1$, for $k \in \{1, \ldots, n\}$ and let $m_0 = m$ represent the mass of the body at the center. The position of the kth body is given by $\mathbf{x}_k = (\cos\theta_k, \sin\theta_k)$ where $\theta_k = 2\pi k/n$ for $k \in \{1, \ldots, n\}$ and by $\mathbf{x}_0 = (0,0)$. It is easy to check that $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ satisfies equation (3) for any value of the central mass m, so it is natural to treat m as a parameter. Recall that r_{ij} represents the distance between the i-th and j-th bodies. Two frequently used formulas are

$$r_{nk}^2 = 2(1-\cos\theta_k) \quad \text{and} \quad r_{nk} = 2\sin(\pi k/n).$$

As mentioned in Section 2, this configuration begets a periodic solution in which the bodies rotate uniformly about the central mass with rotation speed

$\omega = \omega(m)$, which in this case is given by

$$\omega^2 \;=\; m + \tfrac{1}{2}\sigma_n, \qquad \sigma_n \;=\; \sum_{k=1}^{n-1} \frac{1}{r_{nk}} \;=\; \tfrac{1}{2}\sum_{k=1}^{n-1} \csc\frac{\pi k}{n}.$$

This formula follows directly from any component of equation (3). Note that as the mass of the central body increases, the period of the circular orbit decreases.

Recall that $S = D\nabla U(\mathbf{x})$. Direct computation reveals that

$$S = \begin{bmatrix} S_{00} & \cdots & S_{0n} \\ \vdots & & \vdots \\ S_{n0} & \cdots & S_{nn} \end{bmatrix}$$

where S_{ij} is the 2×2 matrix given by

$$\begin{aligned} S_{ij} &= \frac{m_i m_j}{r_{ij}^3}\left[I - 3\mathbf{x}_{ij}\mathbf{x}_{ij}^{\mathrm{T}}\right] \quad \text{if} \quad i \neq j \\ S_{jj} &= -\sum_{i \neq j} S_{ij} \end{aligned}$$

and $\mathbf{x}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{r_{ij}}$ (I is the 2×2 identity matrix). Note that each block S_{ij} is symmetric and that $S_{ij} = S_{ji}$.

Using the fact that the diagonal blocks of S are the negative of the sum of the blocks in the corresponding rows, it is clear that both $\hat{\mathbf{v}} = (1, 0, \ldots, 1, 0)$ and $K\hat{\mathbf{v}} = -(0, 1, \ldots, 0, 1)$ are in the kernel of S. Therefore, $\hat{W} = \operatorname{span}\{\hat{\mathbf{v}}, K\hat{\mathbf{v}}\}$ is a two-dimensional invariant subspace for both $M^{-1}S$ and K. Taking the matrices in equation (6) restricted to $\hat{W}$ yields the 2×2 matrix

$$\begin{bmatrix} \omega^2 - \lambda^2 & -2\omega\lambda \\ 2\omega\lambda & \omega^2 - \lambda^2 \end{bmatrix}.$$

Taking the determinant of this matrix and applying Proposition 2.1 yields the quartic factor $(\lambda^2 + \omega^2)^2$ and the repeated eigenvalues $\pm\omega i, \pm\omega i$. These values are a result of a drift in the center of mass and are evident in any relative equilibrium.

Another two-dimensional invariant subspace which is also evident in any relative equilibrium comes from the configuration itself. Letting $W_0 = \operatorname{span}\{\mathbf{x}, K\mathbf{x}\}$ and applying Proposition 2.1 yields the 2×2 matrix

$$\begin{bmatrix} 3\omega^2 - \lambda^2 & -2\omega\lambda \\ 2\omega\lambda & -\lambda^2 \end{bmatrix}.$$

Here we make use of the fact that $\mathbf{x}$ and $K\mathbf{x}$ are eigenvectors of $M^{-1}S$ with eigenvalues $2\omega^2$ and $-\omega^2$, respectively.[7] Taking the determinant of the matrix above yields the quartic factor $\lambda^2(\lambda^2+\omega^2)$ with eigenvalues $0, 0, \pm\omega i$. These values arise because the equilibrium point $(\mathbf{x}, -\omega MK\mathbf{x})$ of system (2) is not isolated and has two degenerate directions corresponding to rotation or scaling.

Since the factorizations we will be considering lead to even polynomials (Proposition 2.1), we will make the substitution $z = \lambda^2$ and call

$$Q(z) = \det\left[M^{-1}S + (\omega^2 - z)I + 2\sqrt{z}\omega K\right]$$

the *stability polynomial.* It is standard to call a relative equilibrium *nondegenerate* if the remaining $2n-2$ roots of the stability polynomial are nonzero (or the remaining $4n-4$ eigenvalues are nonzero). We will call a relative equilibrium *spectrally stable* if the remaining $2n-2$ roots of the stability polynomial are real and negative, and *linearly stable* if in addition to being spectrally stable, the linearized matrix A in (4) for the remaining $4n-4$ eigenvalues is diagonalizable.

Due to the rotational symmetry of the $1+n$-gon configuration, S has a particularly nice form. After some computation, one can check that the space $U_l = \mathrm{span}\{\mathbf{u}, K\mathbf{u}\}$, where $\mathbf{u} = (\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n)$ and $\mathbf{u}_k = e^{i\theta_{kl}}\mathbf{x}_k$, ($\theta_{kl} = 2\pi kl/n$), form a two-dimensional complex $M^{-1}S$-invariant subspace in $\mathbb{C}^{2n+2}$ for each $l \in \{2, 3, \ldots, [n/2]\}$. When $l = 1$, this perturbation does not leave the central mass fixed at the origin. (This point was overlooked by Maxwell.[3]) Instead of $(0,0)$, the first component of $\mathbf{u}$ must be $\mathbf{u}_0 = (-n/(2m), -in/(2m))$. Note that choosing $l = 0$ yields the two real vectors $\{\mathbf{x}, K\mathbf{x}\}$ already accounted for above.

For $l \geq 2$, the restriction of the operator $M^{-1}S$ to the subspace U_l is

$$\begin{bmatrix} P_l - 3Q_l + 2m & -iR_l \\ iR_l & P_l + 3Q_l - m \end{bmatrix} \tag{7}$$

where

$$P_l = \sum_{k=1}^{n-1} \frac{1 - \cos\theta_k \cos\theta_{kl}}{2r_{nk}^3}, \quad Q_l = \sum_{k=1}^{n-1} \frac{\cos\theta_k - \cos\theta_{kl}}{2r_{nk}^3}, \quad R_l = \sum_{k=1}^{n-1} \frac{\sin\theta_k \sin\theta_{kl}}{2r_{nk}^3}.$$

When $l = 1$, we obtain the matrix

$$\begin{bmatrix} P_1 + 2m + n & -i(R_1 - n) \\ i(R_1 + n/2) & P_1 - m - n/2 \end{bmatrix}. \tag{8}$$

These calculations rely on the fact that a given row of 2×2 blocks of S can be expressed in terms of one block at the end of the row using a rotation matrix. The reader is encouraged to see Moeckel's work for the details.[7]

Taking real and imaginary parts of $\mathbf{u}$ and $K\mathbf{u}$ yields a four-dimensional real invariant subspace for $M^{-1}S$ and K except when n is even and $l = n/2$. In this exceptional case, the vectors $\mathbf{u}$ and $K\mathbf{u}$ are each real, so that we obtain a two-dimensional $M^{-1}S$-invariant subspace. We ignore this case for the moment. Setting $\mathbf{u} = \mathbf{v} + i\mathbf{w}$, yields the four real vectors $\{\mathbf{v}, \mathbf{w}, K\mathbf{v}, K\mathbf{w}\}$ where

$$
\begin{aligned}
\mathbf{v}_k &= \cos\theta_{kl}(\cos\theta_k, \sin\theta_k) \\
\mathbf{w}_k &= \sin\theta_{kl}(\cos\theta_k, \sin\theta_k) \\
(K\mathbf{v})_k &= \cos\theta_{kl}(\sin\theta_k, -\cos\theta_k) \\
(K\mathbf{w})_k &= \sin\theta_{kl}(\sin\theta_k, -\cos\theta_k)
\end{aligned}
$$

are the components for $k \in \{1, 2, \ldots, n\}$ with $(0,0)$ as the first two entries when $l \neq 1$ and $(-n/(2m), 0)$ and $(0, -n/(2m))$ as the first two entries for $\mathbf{v}$ and $\mathbf{w}$, respectively, when $l = 1$. It is easy to check that these vectors are linearly independent. Letting $W_l = \text{span}\{\mathbf{v}, K\mathbf{w}, K\mathbf{v}, \mathbf{w}\}$ for $l \in \{1, 2, \ldots, [n/2]\}$, one can check that the invariant subspaces $\hat{W}, W_0, W_1, \ldots, W_{[n/2]}$ are all M-orthogonal. When n is odd, the union of these spaces has dimension $2 + 2 + 4 \times (n-1)/2 = 2n + 2$, and for n even, we obtain a dimension of $2 + 2 + 4 \times (n-2)/2 + 2 = 2n + 2$. Thus we have completely decomposed $\mathbb{R}^{2n+2}$ into $M^{-1}S$- and K-invariant subspaces from which we can determine all of the eigenvalues for the $1 + n$-gon.

Remark:

1. For a given l, the perturbation corresponding to the invariant subspace W_l pushes the kth body in a direction given by the position of the $k(l+1)$th (mod n) body. For example, when $l = 0$, the perturbation stretches or shrinks the configuration, giving us one of the degenerate directions. As l increases, the perturbation involves more and more twisting requiring a larger and larger central mass for linear stability. We prove this fact in Section 4.

2. The fact that the perturbation for $l = 1$ does not leave the central mass fixed is a result of Proposition 2.1. Since the space $\hat{W}$ is $M^{-1}S$-invariant, so too is its M-orthogonal complement. Thus, if v is a vector in a different invariant subspace, it must belong to the M-orthogonal complement of $\hat{W}$. This means that both the sum of the odd entries of Mv and the even entries of Mv must vanish. If the first two entries for the case $l = 1$ were zero, then these sums would not vanish. Hence the need for the extra factor $-n/(2m)$.

We see from (7) and (8) that the restriction of $W_l = \text{span}\{\mathbf{v}, K\mathbf{w}, K\mathbf{v}, \mathbf{w}\}$ to the matrices in equation (6) is given by the 4×4 matrix

$$\begin{bmatrix} a_l+\omega^2-\lambda^2 & -R_l & -2\lambda\omega & 0 \\ -R_l & b_l+\omega^2-\lambda^2 & 0 & 2\lambda\omega \\ 2\lambda\omega & 0 & b_l+\omega^2-\lambda^2 & R_l \\ 0 & -2\lambda\omega & R_l & a_l+\omega^2-\lambda^2 \end{bmatrix} \tag{9}$$

for $l \geq 2$ where $a_l = P_l - 3Q_l + 2m$ and $b_l = P_l + 3Q_l - m$, and

$$\begin{bmatrix} a_1+\omega^2-\lambda^2 & -(R_1-n) & -2\lambda\omega & 0 \\ -(R_1+n/2) & b_1+\omega^2-\lambda^2 & 0 & 2\lambda\omega \\ 2\lambda\omega & 0 & b_1+\omega^2-\lambda^2 & R_1+n/2 \\ 0 & -2\lambda\omega & R_1-n & a_1+\omega^2-\lambda^2 \end{bmatrix} \tag{10}$$

when $l = 1$ where $a_1 = P_1 + 2m + n$ and $b_1 = P_1 - m - n/2$. Taking the determinant of each matrix yields the quartic factor ($z = \lambda^2$)

$$G_l(z) = (z^2 + \alpha_l z + \beta_l)^2 + 4\omega^2 c_l^2 z$$

where

$$\alpha_l(m) = \begin{cases} m + \sigma_n - 2P_1 - \frac{n}{2} & \text{if } l = 1 \\ m + \sigma_n - 2P_l & \text{if } l \neq 1 \end{cases} \tag{11}$$

$$\beta_l(m) = \begin{cases} 3(P_1 - \frac{n}{2} + \frac{\sigma_n}{2})m + nP_1 + \frac{\sigma_n}{4}(4P_1 + n + \sigma_n) & \text{if } l = 1 \\ 3(P_l + 3Q_l + \frac{\sigma_n}{2})m + (P_l + \frac{\sigma_n}{2})^2 - 9Q_l^2 - R_l^2. & \text{if } l \neq 1 \end{cases} \tag{12}$$

$$c_l = \begin{cases} 2R_1 - \frac{n}{2} & \text{if } l = 1 \\ 2R_l & \text{if } l \neq 1 \end{cases}$$

Here we have used the fact that $\omega^2 = m + \sigma_n/2$. Note that although these formulas seem complicated, both $\alpha_l(m)$ and $\beta_l(m)$ are linear in m.

When n is even and $l = n/2$, we obtain the two-dimensional invariant subspace $W_{\frac{n}{2}} = \text{span}\{\mathbf{u}, K\mathbf{u}\}$ whose restriction to the matrices in equation (6) is given by the 2×2 matrix

$$\begin{bmatrix} a_{\frac{n}{2}}+\omega^2-\lambda^2 & -2\lambda\omega \\ 2\lambda\omega & b_{\frac{n}{2}}+\omega^2-\lambda^2 \end{bmatrix}. \tag{13}$$

Taking the determinant of this matrix yields the quadratic factor ($z = \lambda^2$)

$$F_{\frac{n}{2}}(z) = z^2 + \alpha_{\frac{n}{2}} z + \beta_{\frac{n}{2}}$$

where $\alpha_{\frac{n}{2}}(m)$ and $\beta_{\frac{n}{2}}(m)$ are given by (11) and (12), respectively, with $l = n/2$.

3.2 Properties of P_l, Q_l and R_l

Before we analyze the polynomials G_l and $F_{\frac{n}{2}}$, we state some important facts and identities about the quantities P_l, Q_l and R_l.

(i) $R_l = (Q_{l+1} - Q_{l-1})/2$

(ii) $P_{l+1} - P_l + Q_{l+1} - Q_l = R_l + R_{l+1}$

(iii) P_l and Q_l are strictly increasing in l for $1 \leq l \leq [n/2]$.

(iv) $P_1 = R_1 > 0$, $Q_1 = 0$ and $R_{\frac{n}{2}} = 0$.

(v) P_l, Q_l and R_l are positive for $1 \leq l \leq [n/2]$ except for Q_1 and $R_{\frac{n}{2}}$.

(vi) $P_l > Q_l$ and $P_l \geq R_l$ for $1 \leq l \leq [n/2]$.

The first two items follow from the formulas for the sum and difference of cosine and sine while the third is proved by Moeckel.[7] The fourth item follows straight from the definitions of P_l, Q_l and R_l. The fact that P_l and Q_l are positive (except for Q_1) follows directly from items three and four. When n is odd, we have $R_{\frac{n-1}{2}} = \frac{1}{2}(Q_{\frac{n+1}{2}} - Q_{\frac{n-3}{2}}) = \frac{1}{2}(Q_{\frac{n-1}{2}} - Q_{\frac{n-3}{2}})$ by symmetry. Thus, identity **(i)** and the fact that Q_l is strictly increasing for $1 \leq l \leq [n/2]$ implies that R_l is positive for $1 \leq l \leq [(n-1)/2]$. Finally, the last item follows from the calculations:

$$\begin{aligned} P_l - Q_l &= \sum_{k=1}^{n-1} \frac{(1-\cos\theta_k)(1-\cos\theta_{kl})}{2r_{nk}^3} &= \sum_{k=1}^{n-1} \frac{1-\cos\theta_{kl}}{4r_{nk}} &> 0 \\ P_l - R_l &= \sum_{k=1}^{n-1} \frac{1-\cos\theta_k\cos\theta_{kl} - \sin\theta_k\theta_{kl}}{2r_{nk}^3} &= \sum_{k=1}^{n-1} \frac{1-\cos\theta_{k(l-1)}}{2r_{nk}^3} &\geq 0. \end{aligned}$$

Some important summation formulas which can be derived using standard complex analysis and the formula for summing a finite geometric series are

$$\sum_{k=1}^{n-1} r_{nk} = 2\cot\frac{\pi}{2n}, \quad \sum_{k=1}^{n-1} \cos\theta_k r_{nk} = \cot\frac{3\pi}{2n} - \cot\frac{\pi}{2n}$$

and

$$\sum_{k=1}^{n-1} \cos^2\theta_k r_{nk} = \frac{1}{2}\cot\frac{5\pi}{2n} - \frac{1}{2}\cot\frac{3\pi}{2n} + \cot\frac{\pi}{2n}.$$

These formulas in turn provide convenient expressions for P_l, Q_l and R_l. For example, note that

$$\begin{aligned} P_1 &= \sum_{k=1}^{n-1} \frac{1-\cos^2\theta_k}{2r_{nk}^3} &=& \sum_{k=1}^{n-1} \frac{1+\cos\theta_k}{4r_{nk}} \\ &= \sum_{k=1}^{n-1} \frac{2+(\cos\theta_k - 1)}{4r_{nk}} &=& \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{r_{nk}} - \frac{1}{8}\sum_{k=1}^{n-1} r_{nk} \\ &= \frac{\sigma_n}{2} - \frac{1}{4}\cot\frac{\pi}{2n}. \end{aligned} \tag{14}$$

Similar calculations reveal that

$$P_2 = \frac{5}{4}\sigma_n - \frac{1}{4}\cot\frac{3\pi}{2n} - \frac{3}{4}\cot\frac{\pi}{2n}, \tag{15}$$

$$Q_2 = \frac{3}{4}\sigma_n - \frac{1}{2}\cot\frac{\pi}{2n}, \tag{16}$$

$$R_2 = \sigma_n - \frac{1}{4}\cot\frac{3\pi}{2n} - \frac{3}{4}\cot\frac{\pi}{2n} \quad \text{and} \tag{17}$$

$$P_3 = \frac{5}{2}\sigma_n - \frac{1}{4}\cot\frac{5\pi}{2n} - \frac{3}{4}\cot\frac{3\pi}{2n} - \frac{7}{4}\cot\frac{\pi}{2n}. \tag{18}$$

3.3 Conditions on G_l and $F_{\frac{n}{2}}$ for linear stability

If we fix an n value, the coefficients of the polynomials G_l and $F_{\frac{n}{2}}$ vary with m. We are interested in what values of m make the $1+n$-gon linearly stable. Rather than calculate the roots specifically, it is easier to find necessary and sufficient conditions on the coefficients which yield real and negative roots.

To simplify notation we will let $\gamma_l = \gamma_l(m) = \alpha_l^2 - 4\beta_l$. When n is even and $l = n/2$, γ_l is the discriminant for $F_{\frac{n}{2}}$. Note that the slope of $\alpha_l(m)$ is always one, so that $\gamma_l(m)$ is a parabola opening upward. It is easy to show that $F_{\frac{n}{2}}$ has real, negative and distinct roots if and only if:

$$\alpha_{\frac{n}{2}} > 0, \quad \beta_{\frac{n}{2}} > 0 \quad \text{and} \quad \gamma_{\frac{n}{2}} > 0.$$

Recall that $G_l(z) = (z^2 + \alpha_l z + \beta_l)^2 + 4\omega^2 c_l^2 z$. If $c_l = 0$, then this polynomial would always have repeated roots. Fortunately, this is never the case.

Lemma 3.1 *The coefficient c_l is nonzero for $1 \le l \le [(n-1)/2]$.*

Proof: For $l \neq 1$, $c_l = 2R_l$ which is positive by item **(v)** of the last section. (When n is even and $l = n/2$ we are only interested in the quadratic

$F_{\frac{n}{2}}$ which does not involve $c_{\frac{n}{2}}$.) When $l = 1$, we use the fact that $R_1 = P_1$ and equation (14) to obtain

$$c_1 = \sigma_n - \frac{1}{2} \cot \frac{\pi}{2n} - n/2.$$

We numerically verified that c_1 is negative for $3 \leq n \leq 11$ and positive for $12 \leq n \leq 14$. We can show that $c_1 > 0$ for $n \geq 15$ using some basic estimates. Using the symmetry of the n-gon, and the fact that $\csc x > 1/x$ and $\cot x < 1/x$, we have

$$\begin{aligned}\sigma_n &\geq \csc \frac{\pi}{n} + \csc \frac{2\pi}{n} + \cdots + \csc \frac{7\pi}{n} \\ &> \frac{n}{\pi}(1 + \frac{1}{2} + \cdots + \frac{1}{7}) \\ &> \frac{n}{\pi}(1 + \frac{\pi}{2}) \\ &> \frac{1}{2} \cot \frac{\pi}{2n} + n/2\end{aligned}$$

as desired.

A proof similar to that of Moeckel's[7] shows that G_l has real, negative and distinct roots if and only if

$$\alpha_l > 0, \quad \beta_l \neq 0, \quad \gamma_l \geq 0 \quad \text{and} \quad \delta_l > 0$$

where $\delta_l = \delta_l(m)$ is the discriminant of G_l divided by the positive term $c_l^4 \omega^4$:

$$\delta_l(m) = \beta_l(\alpha_l^2 - 4\beta_l)^2 + c_l^2 \omega^2 \alpha_l (\alpha_l^2 - 36\beta_l) - 27 c_l^4 \omega^4. \tag{19}$$

Note that $\delta_l(m)$ is a fifth degree polynomial in m with leading coefficient identical to the slope of $\beta_l(m)$.

Recall that $G_l(z)$ has repeated roots if and only if $\delta_l = 0$ and $F_{\frac{n}{2}}(z)$ has repeated roots if and only if $\gamma_l = 0$. Suppose that one of these polynomials has a real, negative repeated root and thus the $1+n$-gon has a pair of repeated pure imaginary eigenvalues. In order to have linear stability, we would need a basis of eigenvectors for the linearized matrix A in (4). However, this is never the case.

Lemma 3.2 *Suppose that $F_{\frac{n}{2}}(z)$ or $G_l(z)$ has a real, negative repeated root z_0 corresponding to one of the nondegenerate invariant subspaces W_l described above. Let B' be the restriction of $M^{-1}S + (\omega^2 - z)I + 2\sqrt{z}\omega K$ to W_l. Then the geometric multiplicity of z_0 for B' is always one and consequently, the linearized matrix A in (4) is not diagonalizable.*

Proof: Recall that an eigenvector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ of A with eigenvalue $\lambda_0 = \sqrt{z_0}$ satisfies the equations

$$\begin{aligned} \mathbf{v}_2 &= M(\lambda_0 I - \omega K)\mathbf{v}_1 \\ B\mathbf{v}_1 &= 0 \end{aligned}$$

with B given by (5) evaluated at $\lambda = \lambda_0$. If the geometric multiplicity of z_0 for B' is only one, then there is only a single vector $\mathbf{v}_1$ in the kernel of B, and hence A will have only one eigenvector for the repeated eigenvalue $\lambda_0 = \sqrt{z_0}$.

For $F_{\frac{n}{2}}(z)$, the 2×2 matrix in (13) can never have nullity 2 since the off diagonal terms are always nonzero ($\omega \neq 0$ and $\lambda_0 \neq 0$). For $G_l(z)$, the 4×4 matrices in (9) and (10) will have nullity greater than one if and only if every 3×3 sub-matrix has zero determinant. For $l \neq 1$, taking the determinant of the matrix in (9) with the first row and fourth column deleted gives

$$-2\lambda_0 \omega c_l (b_l + \omega^2 - \lambda_0^2).$$

By Lemma 3.1, this quantity vanishes only if $\lambda_0^2 = b_l + \omega^2$. However, $\lambda_0^2 = z_0 < 0$ by assumption and

$$b_l + \omega^2 \ = \ P_l + 3Q_l + \sigma_n/2 \ > \ 0$$

by item **(v)** of the last section. A similar argument works for $l = 1$. This completes the proof.

The importance of this lemma is that the roots of the polynomials $F_{\frac{n}{2}}(z)$ and $G_l(z)$ must be real, negative *and* distinct in order to have linear stability. In other words, if all the roots of these polynomials are real and negative, but the discriminant vanishes for one of them, then the $1 + n$-gon is spectrally stable, but not linearly stable. We will call a polynomial *stable*, when its roots are real and negative, and call it *linearly stable*, when its roots are real, negative and distinct.

4 Finding the Bifurcation in Stability

For a fixed n, we will show that as the mass m of the central body increases, the polynomials G_l become linearly stable in succession. In other words, as m increases first G_1 becomes linearly stable, then G_2 becomes linearly stable, and so on, until m passes through a bifurcation value h_n where $G_{\frac{n-1}{2}}$ or $F_{\frac{n}{2}}$ become linearly stable depending upon the parity of n.

Since we are interested in the values of the central mass m for which G_l and $F_{\frac{n}{2}}$ are linearly stable, it is important to know when the relevant coefficients $\alpha_l, \beta_l, \gamma_l$, and δ_l are positive. Recall that $\alpha_l, \beta_l, \gamma_l$, and δ_l are functions of m of degree $1, 1, 2$ and 5 respectively. We denote m_{α_l} and m_{β_l} as the lone roots of $\alpha_l(m)$ and $\beta_l(m)$ respectively. Therefore, we have

$$m_{\alpha_l} = \begin{cases} 2P_1 - \sigma_n + \frac{n}{2} & \text{if } l = 1 \\ 2P_l - \sigma_n & \text{if } l \neq 1 \end{cases}$$

$$m_{\beta_l} = \begin{cases} -\dfrac{nP_1 + \frac{\sigma_n}{4}(4P_1 + n + \sigma_n)}{3(P_1 - \frac{n}{2} + \frac{\sigma_n}{2})} & \text{if } l = 1 \\ -\dfrac{(P_l + \frac{\sigma_n}{2})^2 - 9Q_l^2 - R_l^2}{3(P_l + 3Q_l + \frac{\sigma_n}{2})} & \text{if } l \neq 1. \end{cases} \tag{20}$$

To simplify the notation, we define the following coefficients:

$$A_l = \begin{cases} 2\sigma_n + 8P_1 - 5\frac{n}{2} & \text{if } l = 1 \\ 2\sigma_n + 8P_l + 18Q_l & \text{if } l \neq 1 \end{cases}$$

$$B_l = \begin{cases} (2P_1 - \frac{n}{2})^2 - 4\sigma_n(2P_1 + \frac{n}{2}) & \text{if } l = 1 \\ 36Q_l^2 + 4R_l^2 - 8\sigma_n P_l & \text{if } l \neq 1. \end{cases}$$

Then, $\gamma_l(m) = m^2 - 2A_l m + B_l$ and

$$\Gamma_{l\pm} \; = \; A_l \; \pm \; \sqrt{A_l^2 - B_l} \tag{21}$$

are the two roots of the quadratic $\gamma_l(m)$. Since $\delta_l(m)$ is a fifth degree polynomial, it has at least one real root. We will let Δ_l denote the largest real root of $\delta_l(m)$.

The first logical question is whether the roots of $\gamma_l(m)$ are real or not. A short computation shows that

$$A_l^2 - B_l = \begin{cases} 60P_1^2 + 40P_1\sigma_n - 38P_1 n + 4(\sigma_n - n)^2 + 2n^2 & \text{if } l = 1 \\ 64P_l^2 + 40P_l\sigma_n - 4R_l^2 + 4\sigma_n^2 + 72\sigma_n Q_l + 288Q_l(P_l + Q_l) & \text{if } l \neq 1 \end{cases} \tag{22}$$

For $l \neq 1$, items **(v)** and **(vi)** from Section 3.2 show that $A_l^2 - B_l$ is positive. For the case $l = 1$, we can numerically verify that $A_1^2 - B_1 > 0$ for $3 \leq n \leq 10$. To show that $A_1^2 - B_1 > 0$ for $n \geq 11$, we have

$$A_1^2 - B_1 > 60P_1^2 + 40P_1(\sigma_n - n) + 4(\sigma_n - n)^2 + 2n^2,$$

so it suffices to show that $60P_1^2 + 40P_1(\sigma_n - n) > 0$. Dividing by P_1 and substituting the expression in (14) for P_1, this reduces to $\sigma_n > \frac{4}{7}n + \frac{3}{14}\cot\frac{\pi}{2n}$, which is easily verified by an argument similar to the one used in the proof of Lemma 3.1. Therefore, $A_l^2 - B_l$ is always positive and so $\gamma_l(m)$ always has two real roots $m = \Gamma_{l\pm}$. Note that since the leading coefficient of $\gamma_l(m)$ is one, $\gamma_l(m) > 0$ if and only if $m < \Gamma_{l-}$ or $m > \Gamma_{l+}$. Figure 1 gives a rough sketch of the lines $\alpha_l(m), \beta_l(m)$, the parabola $\gamma_l(m)$ and their corresponding roots. Later in this section we explain the relationship between these roots.

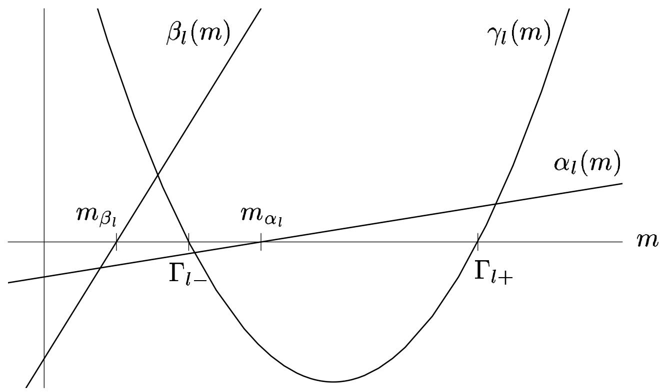

Figure 1. Graph of the coefficients $\alpha_l(m), \beta_l(m)$ and $\gamma_l(m)$.

Lemma 4.1 *The slope of* $\beta_1(m)$, $3(P_1 - n/2 + \sigma_n/2)$, *is positive except for* $3 \leq n \leq 6$.

Proof: We can numerically verify the claim for $3 \leq n \leq 8$. Using (14), it suffices to show that $\sigma_n > \frac{n}{2} + \frac{1}{4}\cot\frac{\pi}{2n}$. An argument similar to the one used in Lemma 3.1 yields

$$\begin{aligned} \sigma_n &\geq \csc\frac{\pi}{n} + \csc\frac{2\pi}{n} + \csc\frac{3\pi}{n} + \csc\frac{4\pi}{n} \quad \text{for } n \geq 9 \\ &> \frac{n}{\pi}(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) \\ &> \frac{n}{\pi}(\frac{\pi+1}{2}) \\ &> \frac{n}{2} + \frac{1}{4}\cot\frac{\pi}{2n} \end{aligned}$$

as desired.

This result is extremely significant in its own right since β_l and δ_l share the same leading coefficient. When $l = 1$, the discriminant $\delta_1(m)$ is positive for sufficiently large m only when $n \geq 7$. Thus, for $3 \leq n \leq 6$, in order to obtain stability we have to hope that in the intervals where $\delta_1(m)$ is positive, the other necessary coefficients are also positive. However, this is never the case.

Theorem 4.2 *For $n = 3, 4, 5, 6$, the $1+n$-gon is not spectrally stable for any value of the central mass m.*

Proof: The relevant constants $m_{\alpha_l}, m_{\beta_l}, \Gamma_{l\pm}$ and Δ_l for each value of n are given in Table 1. (Note that the fourth column gives the mass values for which the $1+n$-gon is degenerate, as $\beta_l = 0$ implies that zero is an eigenvalue. These values agree with those listed in Appendix B of Meyer and Schmidt's[6] paper on the configurations which bifurcate from the $1 + n$-gon.) We handle each case individually.

For $n = 3$, it is necessary that $m > m_{\alpha_1} \approx .6334$ in order for $\alpha_1(m)$ to be positive. However, by Lemma 4.1 and from Table 1, we see that $\delta_1(m)$ is negative for $m > \Delta_1 \approx .2744$. Thus, the polynomial G_1 is never stable for $n = 3$.

For $n = 4$, since Γ_{1-} is negative, it is necessary that $m > \Gamma_{1+} \approx 2.1553$ in order for $\gamma_1(m)$ to be positive. However, by Lemma 4.1 and from Table 1, we see that $\delta_1(m)$ is negative for $m > \Delta_1 \approx 1.0072$. Thus, the polynomial G_1 is never stable for $n = 4$.

For $n = 5$, G_1 is in fact linearly stable in the interval $7.558 \leq m \leq 7.9804$ because all the necessary coefficients are positive (including the discriminant). However, since Γ_{2-} is negative, and $\Gamma_{2+} \approx 45.3904$, it is necessary that $m > 45.3904$ for $\gamma_2(m)$ to be positive. Thus, the polynomials G_1 and G_2 are never stable concurrently.

For $n = 6$, G_1 is in fact linearly stable in the interval $11.5168 \leq m \leq 22.9103$. However, since Γ_{2-} is negative, and $\Gamma_{2+} \approx 70.6461$, it is necessary that $m > 70.6461$ for $\gamma_2(m)$ to be positive. Thus, the polynomials G_1 and G_2 are never stable concurrently.

n	l	m_{α_l}	m_{β_l}	Γ_{l-}	Γ_{l+}	Δ_l
3	1	0.6334	0.7705	-8.8403	0.7685	0.2744
4	1	0.7929	2.3797	-8.8416	2.1553	1.0072
	2	-0.9571	-0.2500	-0.2292	23.7708	
5	1	0.9612	6.4782	-8.7580	4.4805	7.9804
	2	-0.8507	-0.2442	-0.2361	45.3904	48.4064
6	1	1.1340	20.9068	-8.8320	7.7602	22.9103
	2	-0.6160	-0.2201	-0.2179	70.6461	77.5479
	3	-0.0774	0.0060	0.0061	86.8392	

Table 1. The approximate roots of $\alpha_l(m), \beta_l(m), \gamma_l(m)$, and the largest root of $\delta_l(m)$ for the cases $n = 3, 4, 5,$ and 6.

The remainder of the paper will be concerned with the case $n \geq 7$. Note that the slope of $\beta_l(m)$ for $l \neq 1$ is $3(P_l + 3Q_l + \sigma_n/2)$ which is always positive. This fact, along with Lemma 4.1, implies that the leading coefficients of the terms $\alpha_l(m), \beta_l(m), \gamma_l(m)$ and $\delta_l(m)$ are all positive for all l. Hence for sufficiently large m, the $1+n$-gon will be linearly stable. We now show that there is a unique mass value h_n such that $m > h_n$ is a necessary and sufficient condition for linear stability.

The next lemma explains the order in which the important coefficients become positive as m is increased. The sequence goes as follows: We start out with both α_l and β_l negative and γ_l positive. As m increases, β_l becomes positive first, but by the time α_l becomes positive, γ_l has become negative. (So we don't have stability yet.) Then once γ_l becomes positive again, δ_l is now negative. Finally, once m is larger than Δ_l, all four coefficients are positive and will remain that way for all $m > \Delta_l$ (see Figure 2).

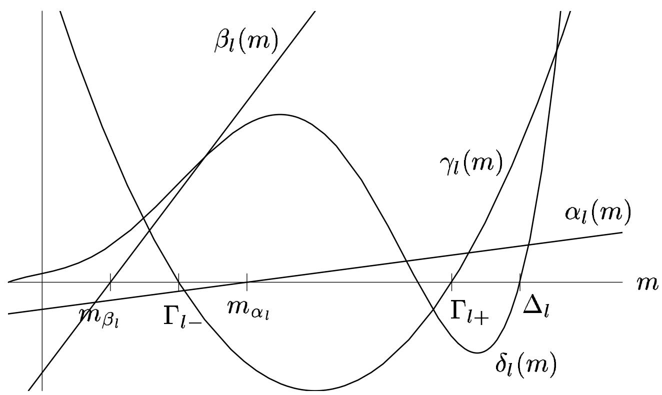

Figure 2. Graph of the coefficients $\alpha_l(m), \beta_l(m)$, $\gamma_l(m)$ and $\delta_l(m)$ illustrating the order of their roots.

Lemma 4.3 *For $n \geq 8$, $m_{\beta_l} < \Gamma_{l-} < m_{\alpha_l} < \Gamma_{l+} < \Delta_l$ for $1 \leq l \leq [n/2]$.*

Proof: We first show that $m_{\beta_l} < m_{\alpha_l}$. For $l = 1$, we have

$$\begin{aligned} m_{\alpha_1} &= 2P_1 + \frac{n}{2} - \sigma_n \;=\; \frac{n}{2} - \frac{1}{2}\cot\frac{\pi}{2n} &> 0 \\ m_{\beta_1} &= -\frac{nP_1 + \frac{\sigma_n}{4}(4P_1 + n + \sigma_n)}{3(P_1 - \frac{n}{2} + \frac{\sigma_n}{2})} &< 0 \end{aligned}$$

where the last inequality follows from Lemma 4.1. For $l \geq 2$, we can verify that $m_{\beta_l} < m_{\alpha_l}$ numerically for $n = 8, \ldots, 13$. For $n \geq 14$, we can rewrite equation (20) as

$$m_{\beta_l} = Q_l - \frac{1}{3}P_l - \frac{1}{6}\sigma_n + \frac{R_l^2}{3P_l + 9Q_l + \frac{3}{2}\sigma_n}.$$

Using the fact that $P_l > R_l$ and Q_l and σ_n are nonnegative, we have

$$m_{\beta_l} < Q_l - \frac{1}{3}P_l - \frac{1}{6}\sigma_n + \frac{1}{3}R_l.$$

Since $m_{\alpha_l} = 2P_l - \sigma_n$, it suffices to show that

$$\frac{7}{3}P_l > Q_l + \frac{5}{6}\sigma_n + \frac{1}{3}R_l. \tag{23}$$

For the case $l = 2$, we can use the expressions from (15), (16) and (17) to obtain

$$\frac{7}{3}P_2 - Q_2 - \frac{5}{6}\sigma_n - \frac{1}{3}R_l \;=\; \sigma_n - \frac{1}{2}\cot\frac{3\pi}{2n} - \cot\frac{\pi}{2n} \;>\; 0$$

by our usual estimates. To prove inequality (23) for the cases $l \geq 3$, it suffices to show that $P_l > 5\sigma_n/6$, since $P_l > Q_l$ and $P_l \geq R_l$. Since for any fixed n, P_l is increasing, it suffices to show that $P_3 > 5\sigma_n/6$. Using equation (18), this is equivalent to

$$\sigma_n > \frac{3}{20}\cot\frac{5\pi}{2n} + \frac{9}{20}\cot\frac{3\pi}{2n} + \frac{21}{20}\cot\frac{\pi}{2n},$$

which follows for $n \geq 14$ from the usual estimates. This finishes the proof that

$$m_{\alpha_l} > m_{\beta_l}. \tag{24}$$

Recall that the leading coefficients of $\alpha_l(m)$, $\beta_l(m)$ and $\gamma_l(m)$ are positive for all l. Simply stated,

$$\alpha_l(m) > 0 \quad \text{iff} \quad m > m_{\alpha_l} \tag{25}$$

$$\beta_l(m) > 0 \quad \text{iff} \quad m > m_{\beta_l} \tag{26}$$

$$\gamma_l(m) > 0 \quad \text{iff} \quad m < \Gamma_{l-} \quad \text{or} \quad m > \Gamma_{l+} \tag{27}$$

(see Figure 1). Therefore, we have

$$\begin{aligned}\gamma_l(m_{\alpha_l}) &= \alpha_l^2(m_{\alpha_l}) - 4\beta_l(m_{\alpha_l}) \\ &= -4\beta_l(m_{\alpha_l}) \\ &< 0\end{aligned}$$

where the inequality follows from (24) and (26). Inequality (27) then implies that $\Gamma_{l-} < m_{\alpha_l} < \Gamma_{l+}$.

Next, we have that

$$\begin{aligned}\gamma_l(m_{\beta_l}) &= \alpha_l^2(m_{\beta_l}) - 4\beta_l(m_{\beta_l}) \\ &= \alpha_l^2(m_{\beta_l}) \\ &> 0\end{aligned}$$

with the strict inequality following from (24) and (25). Therefore, either $m_{\beta_l} < \Gamma_{l-}$ or $m_{\beta_l} > \Gamma_{l+}$. However, the latter inequality implies that $m_{\beta_l} > m_{\alpha_l}$ which contradicts (24). Finally, we note that

$$\begin{aligned}\delta_l(\Gamma_{l+}) &= \beta_l\gamma_l^2 + c_l^2\omega^2\alpha_l(\gamma_l - 32\beta_l) - 27c_l^4\omega^4 \\ &= -32c_l^2\omega^2\alpha_l\beta_l - 27c_l^4\omega^4 \\ &< 0\end{aligned}$$

where α_l, β_l and γ_l are evaluated at $m = \Gamma_{l+}$. Thus since $\delta_l(m)$ is of odd degree with leading coefficient positive, its largest root, Δ_l, must be greater than Γ_{l+}. This completes the proof.

Lemma 4.4 *For each $n \geq 7$ and for each integer l, $1 \leq l < n/2$, the polynomial G_l is linearly stable if and only if $m > \Delta_l$. If n is even, then $F_{\frac{n}{2}}$ is linearly stable if and only if $m > \Gamma_{\frac{n}{2}+}$.*

Proof: The second statement follows directly from Lemma 4.3. In order for $F_{\frac{n}{2}}$ to be linearly stable, $\alpha_{\frac{n}{2}}, \beta_{\frac{n}{2}}$ and $\gamma_{\frac{n}{2}}$ must all be positive. This occurs if and only if $m > \Gamma_{\frac{n}{2}+}$. For the quartic G_l, we first handle the case $n = 7$ separately. (This is necessary because Lemma 4.3 is not valid when $n = 7$ and $l = 2$.) We list the relevant data in Table 2. It is clear from the data in all three cases that G_l is linearly stable when $m > \Delta_l$. To argue this is also a necessary condition, one can calculate that $\delta_l(\Gamma_{l+}) < 0$ and that Δ_l is the only root of $\delta_l(m)$ larger than Γ_{l+}. Thus the only way to have all four coefficients positive is when $m > \Delta_l$. This finishes the case $n = 7$. Note that the last column of Table 2 is increasing, and that the $1 + 7$-gon is linearly stable if and only if $m > 139.8523$.

l	m_{α_l}	m_{β_l}	Γ_{l-}	Γ_{l+}	Δ_l
1	1.3094	-643.2843	-9.0892	11.8784	15.7260
2	-0.2846	-0.1814	-0.1813	98.8129	110.2791
3	0.9819	0.3242	0.3274	135.5674	139.8523

Table 2. The (approximate) roots of $\alpha_l(m), \beta_l(m), \gamma_l(m)$, and the largest root of $\delta_l(m)$ for the case $n = 7$.

To do the cases $n \geq 8$, Lemma 4.3 shows that it is sufficient for $m > \Delta_l$ as then all the relevant coefficients will be positive. For the other direction, we know that α_l, β_l, and γ_l are positive concurrently if and only if $m > \Gamma_{l+}$. To show it is necessary that $m > \Delta_l$, we need only show that Δ_l is the only root of $\delta_l(m)$ larger than Γ_{l+}. Recall that (19)

$$\delta_l = \beta_l\gamma_l^2 + c_l^2(m + \frac{\sigma_n}{2})\alpha_l\gamma_l - 32c_l^2(m + \frac{\sigma_n}{2})\alpha_l\beta_l - 27c_l^4(m + \frac{\sigma_n}{2})^2.$$

Differentiating this expression with respect to m and evaluating at $m = \Gamma_{l+}$ yields

$$\delta_l'(\Gamma_{l+}) = -24c_l^2\mu\beta_l - 36c_l^2\mu\alpha_l\beta_l' - 32c_l^2\alpha_l\beta_l - 54c_l^4\mu$$

where α_l and β_l are evaluated at $m = \Gamma_{l+}$ and $\mu = \Gamma_{l+} + \sigma_n/2$. (Here we use the fact that $\alpha_l^2(\Gamma_{l+}) = 4\beta_l(\Gamma_{l+})$.) Lemma 4.3 gives $\alpha_l(\Gamma_{l+}) > 0$ and $\beta_l(\Gamma_{l+}) > 0$ so that the above quantity is negative. Thus, not only is the discriminant negative at $m = \Gamma_{l+}$, but it is also decreasing. Further computation and estimation reveals that the third derivative of γ_l with respect to m is positive for $m \geq \Gamma_{l+}$. This in turn implies that $\delta_l'(m)$ is concave up for all $m \geq \Gamma_{l+}$ and therefore $\delta_l(m)$ has exactly one root (Δ_l) larger than Γ_{l+}. This completes the proof of the lemma.

As the largest root of the fifth-degree polynomial $\delta_l(m)$, it is unlikely that an explicit formula for Δ_l exists, not to mention the fact that it would probably be incomprehensible. Thus it is reasonable to seek an upper bound for Δ_l which has a useful explicit formula.

Recall that $G_l(z) = (z^2 + \alpha_l z + \beta_l)^2 + 4\omega^2 c_l^2 z$. From Lemma 4.3 we know that when $m > \Gamma_{l+}$, α_l, β_l and γ_l are positive. This means that the polynomial $z^2 + \alpha_l z + \beta_l$ has two real and distinct negative roots r_1 and r_2 with $r_1 < -\alpha_l/2 < r_2 < 0$. Since $G_l(r_{1,2}) = 4\omega^2 c_l^2 r_{1,2} < 0$ and $G_l(z) > 0$ for $z > 0$, the Intermediate Value Theorem tells us that G_l has two negative real roots s_1 and s_2 satisfying $s_1 < r_1 < -\alpha_l/2 < r_2 < s_2 < 0$. When $\Gamma_{l+} < m < \Delta_l$, we know from Lemma 4.3 and from the proof of Lemma 4.4, that the discriminant of $G_l(z)$ is negative. This means that s_1 and s_2 are the only real roots of $G_l(z)$. When $m = \Delta_l$, the discriminant of G_l vanishes, and a negative real, repeated root is born in addition to s_1 and s_2. Finally, as m increases past Δ_l, $G_l(-\alpha_l/2)$ eventually becomes positive and G_l has four distinct negative real roots. That is, G_l will be linearly stable (see Figure 3).

Finding a sufficient condition for $G_l(-\alpha_l/2) > 0$ leads to the value $m = \Lambda_l$ given by

$$\Lambda_l = A_l + 2\sqrt{2}|c_l| + \sqrt{(A_l + 2\sqrt{2}|c_l|)^2 - B_l + 2\sqrt{2}|c_l|\sigma_n}\ .$$

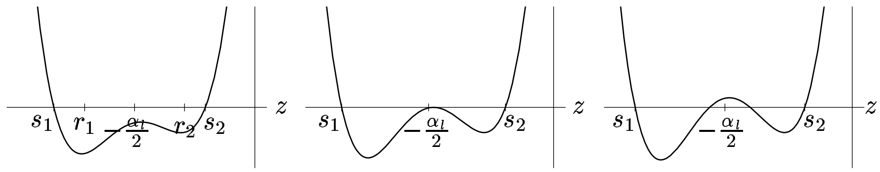

Figure 3. Graph of the quartic $G_l(z)$ for specific m-values from the regimes $\Gamma_{l+} < m < \Delta_l$, $m = \Delta_l$ and $m > \Delta_l$, respectively.

It is a straight-forward argument to show that A_l is positive for all l from which it follows that Λ_l is real and $\Lambda_l > \Gamma_{l+}$. For the rest of the paper we will only be concerned with the larger root of $\gamma_l(m)$ so that Γ_l is understood to represent Γ_{l+}.

Lemma 4.5 *For $n \geq 7$, if $m > \Lambda_l$, then G_l is linearly stable. Moreover,*

$$\Gamma_l < \Delta_l < \Lambda_l < \Gamma_{l+1} \quad \textit{for} \quad 1 \leq l \leq [(n-1)/2].$$

Proof: Since $\Lambda_l > \Gamma_l$, the coefficients α_l, β_l, and γ_l are all positive for $m > \Lambda_l$. So by our arguments above, it suffices to show that $m > \Lambda_l$ implies that $G_l(-\alpha/2) > 0$. We compute that

$$G_l(-\alpha_l/2) = \frac{1}{16}\gamma_l^2 - 2\omega^2 c_l^2 \alpha_l,$$

which is positive if and only if

$$\gamma_l^2 - 32c_l^2\omega^4 \;>\; \begin{cases} 32c_1^2\omega^2(\sigma_n/2 - 2P_1 - n/2) & \text{if } l = 1 \\ 32c_l^2\omega^2(\sigma_n/2 - 2P_l) & \text{if } l \neq 1. \end{cases}$$

It is not difficult to see that the quantities on the right-hand side of the above inequality are negative. Therefore, to show that $G_l(-\alpha_l/2) > 0$, it suffices to show that $\gamma_l^2 - 32c_l^2\omega^4 > 0$ or equivalently,

$$m^2 - 2(A_l + 2\sqrt{2}|c_l|)m + B_l - 2\sqrt{2}|c_l|\sigma_n \;>\; 0.$$

The largest root of the quadratic in this inequality is $m = \Lambda_l$. Thus we have shown that for $m > \Lambda_l$, $G_l(-\alpha_l/2) > 0$ which implies that G_l is linearly stable. This proves the first statement of the lemma and shows that $\Delta_l > \Lambda_l$.

The only inequality left unproven is $\Lambda_l < \Gamma_{l+1}$. This is significant, as it relates the stability of G_l with that of G_{l+1}. To prove this, we make use of identity **(ii)** from Section 3.2:

$$P_{l+1} - P_l + Q_{l+1} - Q_l \;=\; R_l + R_{l+1}. \tag{28}$$

We explain the case $l \geq 2$ (the case $l = 1$ is similar). Note that $|c_l| = 2R_l$ since R_l is nonnegative. We have from equation (28) that

$$\begin{aligned} A_{l+1} &= 2\sigma_n + 8P_l + 10Q_{l+1} + 8Q_l + 8(R_l + R_{l+1}) \\ &> 2\sigma_n + 8P_l + 18Q_l + 4\sqrt{2}R_l \\ &= A_l + 4\sqrt{2}R_l \end{aligned}$$

where the middle inequality follows from the fact that R_l is nonnegative and that Q_l is strictly increasing in l. Next, equation (28) also gives us

$$A_{l+1}^2 - B_{l+1} > \text{ radicand of } \Lambda_l$$

after some computation. Since $A_{l+1} > A_l + 2\sqrt{2}|c_l|$ and the radicand of Γ_{l+1} is greater than the radicand of Λ_l, it follows that $\Gamma_{l+1} > \Lambda_l$.

Lemma 4.5 not only provides us with an upper bound on the seemingly elusive quantity Δ_l, it also provides us with the bifurcation value for the $1 + n$-gon relative equilibrium. Since $\Lambda_l < \Gamma_{l+1} < \Delta_{l+1}$, it is clear that as m increases, G_l will become linearly stable before G_{l+1} does. This proves that the polynomials G_l become linearly stable in succession. In some sense, this means that the more the configuration is twisted, the more unstable it becomes, or conversely, the most stable perturbations involve less rotation of the n-gon (including the one that moves the central mass). If n is even, then since $\Lambda_{\frac{n-2}{2}} < \Gamma_{\frac{n}{2}}$, $F_{\frac{n}{2}}$ is the last polynomial to become linearly stable and this happens when $m > \Gamma_{\frac{n}{2}}$. If n is odd, then $G_{\frac{n-1}{2}}$ is the last polynomial to become linearly stable and this happens when $m > \Delta_{\frac{n-1}{2}}$. For n even, we have a precise formula for the bifurcation value while for n odd, we have explicit expressions bounding the bifurcation value above and below. This completes the proof of our main result.

Theorem 4.6 *For n even, the $1+n$-gon relative equilibrium is linearly stable if and only if $m > \Gamma_{\frac{n}{2}}$. For n odd, the $1+n$-gon relative equilibrium is linearly stable if and only if $m > \Delta_{\frac{n-1}{2}}$.*

5 Estimating the Bifurcation Value

As explained in the last section, we have an explicit formula for the bifurcation value given by $m = \Gamma_{\frac{n}{2}}$ for n even. When n is odd, the best we can do is bound the bifurcation value below by $\Gamma_{\frac{n-1}{2}}$ and above by $\Lambda_{\frac{n-1}{2}}$. Fortunately, these two bounds are asymptotic to each other as n approaches infinity. Table 3 lists the bifurcation value h_n for several different values of n. In this section

n	h_n	$h_n/\tau n^3$	n	h_n	$h_n/\tau n^3$
7	139.8523	0.9372	15	1,443.300	0.9830
8	212.2611	0.9530	20	3,446.298	0.9902
9	304.1366	0.9590	50	54,274.97	0.9981
10	420.9930	0.9677	100	434,797.6	0.9995
11	562.1966	0.9709	500	5.43780323×10^7	1.0000
12	733.9608	0.9763	1,000	4.350332026×10^8	1.0000
13	934.9493	0.9782	10,000	$4.350365372 \times 10^{11}$	1.0000
14	1,172.075	0.9819	100,000	$4.350365691 \times 10^{14}$	1.0000

Table 3. The bifurcation value h_n, where the $1+n$-gon becomes linearly stable, for various values of n and its comparison with the estimate τn^3.

we prove that h_n increases asymptotically to τn^3, where τ is given by the formula:

$$\tau = \frac{13+4\sqrt{10}}{2\pi^3} \cdot \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}.$$

Using a computer, one can calculate that $\tau = 0.435036581297$ is accurate to twelve decimal places. Remarkably, Maxwell calculated τ to be 0.4352 in his work in 1859.[4] He achieves this by correctly "guessing" the force (only in the case with n even) which will produce the most instability, although he provides no proof for why this is the case. Still, it is a great tribute to his insight and ability that Maxwell was able to arrive at this result so well ahead of his time.

We handle the case when n is even first. Calculations show that

$$\begin{aligned} P_{\frac{n}{2}} = \sum_{k=1}^{n-1} \frac{1-\cos\theta_k \cos\theta_{k\frac{n}{2}}}{2r_{nk}^3} &= \sum_{k=1}^{n-1} \frac{1-\cos\theta_k}{2r_{nk}^3} + \sum_{k\,\mathrm{odd}}^{n-1} \frac{2\cos\theta_k}{2r_{nk}^3} \\ &= \frac{\sigma_n}{4} + \sum_{k\,\mathrm{odd}}^{n-1} \frac{1}{r_{nk}^3} - \frac{1}{2}\sum_{k\,\mathrm{odd}}^{n-1} \frac{1}{r_{nk}}. \end{aligned}$$

To ease on notation, we define the sums (for n even)

$$\xi_n = \sum_{k\,\mathrm{odd}}^{n-1} \frac{1}{r_{nk}^3} \quad \text{and} \quad \eta_n = \sum_{k\,\mathrm{odd}}^{n-1} \frac{1}{r_{nk}}.$$

A similar calculation yields $Q_{\frac{n}{2}} = -\sigma_n/4 + \xi_n$. Plugging these expressions for

$P_{\frac{n}{2}}$ and $Q_{\frac{n}{2}}$ into formulas (21) and (22) yields

$$\Gamma_{\frac{n}{2}} = 26\xi_n - 4\eta_n - \frac{\sigma_n}{2} + 4\sqrt{(5\xi_n - \eta_n)(8\xi_n - \eta_n)}.$$

We will show that ξ_n is order n^3 while η_n and σ_n are order $n \ln n$. Hence, $\Gamma_{\frac{n}{2}} \sim (26 + 8\sqrt{10})\xi_n$.

Lemma 5.1 *ξ_n is asymptotic to $\hat{\tau} n^3$ where*

$$\hat{\tau} = \frac{1}{4\pi^3} \cdot \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}.$$

Proof: We will prove the case when $n/2$ is even. The argument for $n/2$ odd is virtually identical. Using the symmetry of the n-gon and the fact that $r_{nk} = 2\sin(\frac{\pi k}{n})$, we have

$$\xi_n = \frac{1}{4}\left(\csc^3 \frac{\pi}{n} + \csc^3 \frac{3\pi}{n} + \cdots + \csc^3 \frac{(n/2-1)\pi}{n}\right). \tag{29}$$

We will show that $\lim_{n\to\infty} \xi_n/n^3 = \hat{\tau}$. Using the fact that $\csc x > 1/x$, it is easy to see that

$$\frac{\xi_n}{n^3} > \frac{1}{4\pi^3}\left(1 + \frac{1}{3^3} + \ldots + \frac{1}{(n/2-1)^3}\right). \tag{30}$$

Using the Taylor expansion of $\sin^3 x$ yields $x^3 - \frac{1}{2}x^5 < \sin^3 x$ and therefore

$$\csc^3 x < \frac{1}{x^3 - \frac{1}{2}x^5} = \frac{1}{x^3} + \frac{1}{x(2-x^2)} \tag{31}$$

provided $x < \sqrt{2}$. In order to make use of this inequality, we will split the terms in ξ_n into two groups. Let k_n be the largest odd integer less than $n/6$. For the terms with $k > k_n$, $\csc \frac{k\pi}{n} < 2$. For any $k \le k_n$, we may apply inequality (31) to obtain

$$\frac{\csc^3(\frac{k\pi}{n})}{n^3} < \frac{1}{k^3\pi^3} + \frac{1}{k\pi(2n^2 - k^2\pi^2)}.$$

Letting $f(u) = 1/(u\pi(2n^2 - u^2\pi^2))$, it is clear that $f(u)$ is positive and decreasing for $1 \le u \le k_n$. Since there are exactly $n/4$ terms in (29), we have that

$$\begin{aligned}\frac{\xi_n}{n^3} &< \frac{1}{4\pi^3}\left(1 + \frac{1}{3^3} + \ldots + \frac{1}{(k_n)^3}\right) + \frac{1}{4}\cdot\frac{n}{4}\left(f(1) + \frac{8}{n^3}\right)\\ &< \frac{1}{4\pi^3}\left(1 + \frac{1}{3^3} + \ldots + \frac{1}{(n/2-1)^3}\right) + \frac{n}{16\pi(2n^2 - \pi^2)} + \frac{1}{2n^2}.\end{aligned} \tag{32}$$

Letting

$$\hat{\tau}_n = \frac{1}{4\pi^3}\left(1 + \frac{1}{3^3} + \ldots + \frac{1}{(n/2-1)^3}\right),$$

inequalities (30) and (32) give

$$\hat{\tau}_n \;<\; \frac{\xi_n}{n^3} \;<\; \hat{\tau}_n + \frac{n}{16\pi(2n^2 - \pi^2)} + \frac{1}{2n^2}.$$

Letting n go to infinity yields the desired result.

Lemma 5.2 *σ_n and η_n are asymptotic to $\frac{n}{\pi}\ln n$ and $\frac{n}{2\pi}\ln n$ respectively.*

Proof: We verify the result for σ_n first. Again using the symmetry of the n-gon, we have

$$\sigma_n = \sum_{k=1}^{\frac{n}{2}-1} \csc\frac{\pi k}{n} + \frac{1}{2}$$

when n is even. Since the function $g(u) = \csc\frac{\pi u}{n}$ is decreasing for $1 \le u \le n/2$, we have

$$g(2) + g(3) + \cdots + g(n/2) \;<\; \int_1^{n/2} g(u)\,du \;<\; g(1) + g(2) + \cdots + g(n/2-1)$$

using lower and upper approximating sums. It follows that

$$\int_1^{n/2} g(u)\,du + \frac{1}{2} \;<\; \sigma_n \;<\; g(1) + \int_1^{n/2} g(u)\,du$$

which evaluates to

$$\frac{n}{\pi}\ln\left(\cot\frac{\pi}{2n}\right) + \frac{1}{2} \;<\; \sigma_n \;<\; \csc\frac{\pi}{n} + \frac{n}{\pi}\ln\left(\cot\frac{\pi}{2n}\right).$$

From this inequality it follows that

$$\lim_{n\to\infty} \frac{\sigma_n}{(n\ln n)/\pi} = 1.$$

The same argument works for η_n by using $h(u) = \csc\frac{\pi}{n}(2u-1)$ in place of $g(u)$ and carrying out the same approximations. The $1/2$ factor arises from the integration of $h(u)$ and is expected since η_n contains half of the terms of σ_n. This completes the proof of the lemma.

The case when n is odd is slightly more complicated since we do not have an explicit expression for the bifurcation value and the formulas we do have are more involved. We will let c denote the crucial index $(n-1)/2$. Recall that for n odd, $\Gamma_c < h_n < \Lambda_c$. Since $\theta_{kc} = \pi k - \pi k/n$, we have

$$\cos\theta_{kc} = \begin{cases} \cos\frac{\pi k}{n} & k \text{ even} \\ -\cos\frac{\pi k}{n} & k \text{ odd} \end{cases}$$

and

$$\sin\theta_{kc} = \begin{cases} -\sin\frac{\pi k}{n} & k \text{ even} \\ \sin\frac{\pi k}{n} & k \text{ odd.} \end{cases}$$

Using these formulas, the symmetry of the n-gon, and $r_{nk} = 2\sin\frac{\pi k}{n}$, we see that

$$\begin{aligned} P_c &= \sum_{k=1}^{c} \frac{1}{r_{nk}^3} + \sum_{k=1}^{c} (-1)^{k+1} \frac{\cos\frac{2\pi k}{n}\cos\frac{\pi k}{n}}{r_{nk}^3} \\ &= \frac{1}{8}\sum_{k=1}^{c} \csc^3\frac{\pi k}{n} + \frac{1}{8}\sum_{k=1}^{c} (-1)^{k+1}\cot\frac{\pi k}{n}\csc^2\frac{\pi k}{n} - \frac{1}{4}\sum_{k=1}^{c} (-1)^{k+1}\cot\frac{\pi k}{n} \end{aligned}$$

An argument similar to the one used in the proof of Lemma 5.2 shows that the last term in the sum above is order $n\ln n$. Likewise, an argument similar to the one used in the proof of Lemma 5.1 shows that

$$\sum_{k=1}^{c} \csc^3\frac{\pi k}{n} \sim \frac{n^3}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3} \qquad \text{and}$$

$$\sum_{k=1}^{c} (-1)^{k+1}\cot\frac{\pi k}{n}\csc^2\frac{\pi k}{n} \sim \frac{n^3}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}.$$

It follows that P_c is asymptotic to $\hat{\tau} n^3$.

Further computation reveals that

$$P_c - Q_c = \frac{\sigma_n}{4} + \frac{1}{4}\sum_{k=1}^{c} (-1)^{k+1}\cot\frac{\pi k}{n}.$$

Since the right-hand side is order $n\ln n$, it follows that Q_c is asymptotic to P_c. Finally, we have that

$$\begin{aligned} R_c &= \sum_{k=1}^{c} (-1)^{k+1}\frac{\sin\frac{2\pi k}{n}\sin\frac{\pi k}{n}}{r_{nk}^3} \\ &= \frac{1}{4}\sum_{k=1}^{c} (-1)^{k+1}\cot\frac{\pi k}{n} \end{aligned}$$

which means that R_c is lower order than P_c and Q_c. Since

$$\Lambda_c = A_c + 4\sqrt{2}R_c + \sqrt{(A_c + 4\sqrt{2}R_c)^2 - B_c + 4\sqrt{2}R_c\sigma_n},$$

we see that $\Gamma_c \sim \Lambda_c$. Using the fact that $P_c \sim Q_c$, equations (21) and (22) imply that

$$h_n \ \sim\ \Gamma_c \ \sim\ (26 + 8\sqrt{10})\,\hat{\tau}\,n^3.$$

We have proven the following theorem:

Theorem 5.3 *The $1+n$-gon relative equilibrium becomes linearly stable at a central mass value which is asymptotic to τn^3 where*

$$\tau = \frac{13 + 4\sqrt{10}}{2\pi^3} \cdot \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}.$$

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