Spectral Instability of Relative Equilibria in the Planar N-Body Problem

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October 16, 1998

Abstract

An inequality is derived which must be satisfied for a relative equilibrium of the n-body problem to be spectrally stable. This inequality is studied in the equal mass case and using simple geometric estimates, it is shown that any relative equilibrium of n equal masses is not spectrally stable for $n \geq 24,306$.

MSC Codes: 70F10, 70F15

1 Introduction

Relative equilibria in the Newtonian n-body problem are special planar periodic orbits of masses which rotate rigidly about their center of mass. Locating a relative equilibrium means finding a configuration in which the forces due to gravity are perfectly balanced by the centrifugal force [9]. This involves finding the critical points of the Newtonian gradient potential restricted to a mass ellipsoid by solving a system of highly nonlinear equations (see Section 2.1). The first such examples were discovered by Euler [3] (the collinear solutions in the 3-body problem) and Lagrange [4] (the equilateral triangle solutions in the 3-body problem). Other well known examples include the regular n-gon (n equal masses placed at the vertices of a regular n-gon) and the $1 + n$-gon (the n-gon configuration with a central body of arbitrary mass m). Albouy recently showed that there are exactly four types of relative equilibria for four equal masses: a collinear solution, a square, an equilateral triangle with a body at the center and an isosceles triangle with a body on the axis of symmetry [1]. Surprisingly, there is still little known about relative equilibria in general. For example, it is unknown whether the number of relative equilibrium equivalence classes for 4 unequal masses is finite [13].

Given a relative equilibrium, it is natural to investigate its stability as a means of determining the behavior of nearby solutions. This entails linearizing the differential equation about the equilibrium and analyzing the associated linear system. Unfortunately, this is a difficult task in practice as it involves computing the eigenvalues of a complicated $4n$ by $4n$ matrix. Since we are dealing with a Hamiltonian system, the characteristic polynomial is even and consequently, equilibrium points are never asymptotically stable [8]. In general, an equilibrium is called *spectrally stable* if its eigenvalues are all pure imaginary and *linearly stable* if, in addition, the associated matrix is diagonalizable.

For the equilateral triangle relative equilibrium, Routh derived a stability inequality which depends on the masses of the three bodies [12]. This inequality is satisfied only if one of the masses is much larger than the other two, a situation which occurs in our solar system with Jupiter, the Sun and the Trojan asteroids, the Sun being the dominant mass. Similarly, Moeckel shows in [9] that when $n \geq 7$, the $1+n$-gon is linearly stable provided the central mass is large enough. In contrast, the regular n-gon (with n equal masses) is not spectrally stable for any n [9].

Based on this evidence, Moeckel conjectured that any spectrally (and linearly) stable relative equilibrium must have a dominant mass. If true, this conjecture implies that any relative equilibrium consisting of all equal masses should be unstable. Examining the next to leading coefficient of the characteristic polynomial for a relative equilibrium, we derive an inequality which must be satisfied for spectral stability. After applying this inequality to some specific examples, we use it to prove that any relative equilibrium of n equal masses is not spectrally stable, provided $n \geq 24{,}306$. Our proof relies on geometric estimates and counting arguments, succeeding without knowing the actual location of the bodies.

2 Determining Stability of Relative Equilibria

In this section, we derive the equations for a relative equilibrium and give a precise definition for its spectral and linear stability. The characteristic polynomial is discussed and a necessary condition for the spectral stability of a relative equilibrium is derived.

2.1 Relative equilibria in the planar n-body problem

We let the mass and position of the n bodies be given by m_i and $\mathbf{q}_i \in \mathbb{R}^2$, $i = 1, \ldots, n$ respectively. Let $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ be the distance between the ith and jth bodies and let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in \mathbb{R}^{2n}$. Using Newton's law of motion and the inverse square law for attraction due to gravity, the second-order equation for the ith body is given by

$$m_i \ddot{\mathbf{q}}_i \;=\; \sum_{i \neq j} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} \;=\; \frac{\partial U}{\partial \mathbf{q}_i},$$

where $U(\mathbf{q})$ is the Newtonian potential function:

$$U(\mathbf{q}) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}.$$

We let the momenta of each body be $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$ and let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{2n}$. The equations of motion can then be written as

$$\begin{aligned} \dot{\mathbf{q}} &= M^{-1}\mathbf{p} = \frac{\partial H}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} &= \nabla U(\mathbf{q}) = -\frac{\partial H}{\partial \mathbf{q}} \end{aligned} \tag{1}$$

where M is the diagonal mass matrix with diagonal $m_1, m_1, m_2, m_2, \ldots, m_n, m_n$ and $H(\mathbf{q}, \mathbf{p})$ is the Hamiltonian function:

$$H(\mathbf{q}, \mathbf{p}) \;=\; \sum_{i=1}^{n} \frac{\|\mathbf{p}_i\|^2}{2m_i} - U(\mathbf{q}) \;=\; \frac{1}{2}\mathbf{p}^{\mathrm{T}} M^{-1} \mathbf{p} - U(\mathbf{q}).$$

Next, let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ so that $e^{\omega J t} = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}$. To introduce coordinates that uniformly rotate about the origin with period $2\pi/\omega$, we let $\mathbf{x}_i = e^{\omega J t}\mathbf{q}_i$ and $\mathbf{y}_i = e^{\omega J t}\mathbf{p}_i$. This is a symplectic change of variables preserving the Hamiltonian structure of system (1) [8]. The new system becomes

$$\begin{aligned}\dot{\mathbf{x}} &= \omega K\mathbf{x} + M^{-1}\mathbf{y} &= \frac{\partial \hat{H}}{\partial \mathbf{y}} \\ \dot{\mathbf{y}} &= \nabla U(\mathbf{x}) + \omega K\mathbf{y} &= -\frac{\partial \hat{H}}{\partial \mathbf{x}}\end{aligned} \tag{2}$$

where K is a $2n \times 2n$ block diagonal matrix with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ on the diagonal and $\hat{H}(\mathbf{x}, \mathbf{y})$ is the Hamiltonian function:

$$\hat{H}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{T}} M^{-1}\mathbf{y} - U(\mathbf{x}) - \omega \mathbf{x}^{\mathrm{T}} K\mathbf{y}.$$

The new term added to the Hamiltonian, $\mathbf{x}^{\mathrm{T}} K\mathbf{y}$, is often referred to as the Coriolis force [8]. Using the fact that $KM = MK$ and $K^2 = -I$, an equilibrium $(\mathbf{x}, \mathbf{y})$ of system (2) must satisfy $\mathbf{y} = -\omega MK\mathbf{x}$ and

$$\nabla U(\mathbf{x}) + \omega^2 M\mathbf{x} = 0. \tag{3}$$

A *relative equilibrium* of the n-body problem is a configuration $\mathbf{x} \in \mathbb{R}^{2n}$ which satisfies the algebraic equations in (3) for some value of ω (see [8], [9] or [10]). The ith component in (3) is given by

$$-\omega^2 m_i \mathbf{x}_i = \sum_{i \neq j} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^3}. \tag{4}$$

Note that if $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is a relative equilibrium, then

$$\begin{aligned} c\mathbf{x} &= (c\mathbf{x}_1, c\mathbf{x}_2, \ldots, c\mathbf{x}_n) \quad \text{and} \\ R\mathbf{x} &= (R\mathbf{x}_1, R\mathbf{x}_2, \ldots, R\mathbf{x}_n) \end{aligned}$$

are also relative equilibria for any constant c and any $R \in SO(2)$. When counting relative equilibria it is standard to fix the scaling (a unique value of c) and identify any configurations which are rotationally equivalent via the equivalence relation $\mathbf{x} \sim R\mathbf{x}$ for $R \in SO(2)$. If we sum all the components of equations (4), then the summation terms on the right-hand sides cancel out and we have

$$\sum_{i=1}^{n} m_i \mathbf{x}_i = 0. \tag{5}$$

In other words, the center of mass of any relative equilibrium is the center of rotation, which in our set up is the origin.

Let $I(\mathbf{x})$ denote the moment of inertia, that is,

$$I(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n} m_i \|\mathbf{x}_i\|^2.$$

We can then write equation (3) as

$$\nabla U(\mathbf{x}) + \omega^2 \nabla I(\mathbf{x}) = 0 \tag{6}$$

In other words, relative equilibria are critical points of the function U restricted to the "mass ellipsoid" defined as the level set $I = c$, where ω^2 plays the role of a Lagrange multiplier. If we take the dot product of a relative equilibrium $\mathbf{x}$ with equation (6) and use the fact that U is homogeneous of degree -1 and I is homogeneous of degree 2, then we obtain $-U(\mathbf{x}) + 2\,\omega^2 I(\mathbf{x}) = 0$ or

$$\omega^2 = \frac{U(\mathbf{x})}{2I(\mathbf{x})}. \tag{7}$$

This expression will be useful later.

2.2 Spectral and linear stability

Linearizing system (2) about the relative equilibrium $(\mathbf{x}, \mathbf{y})$ yields the matrix

$$A = \begin{bmatrix} \omega K & M^{-1} \\ S & \omega K \end{bmatrix} \tag{8}$$

where $S = D\nabla U(\mathbf{x})$ is a $2n \times 2n$ symmetric matrix. We will let $P(\lambda)$ denote the characteristic polynomial of A. $P(\lambda)$ is of degree $4n$ and is an even polynomial, since A is a Hamiltonian matrix [8].

There are two identifiable subspaces which are invariant under A, arising from the degeneracies and integrals of the problem [10]. The subspace W_1 of $\mathbb{C}^{4n}$ spanned by the four vectors

$$(\mathbf{x}, 0),\ (0, M\mathbf{x}),\ (K\mathbf{x}, 0),\ (0, KM\mathbf{x})$$

is invariant under A with eigenvalues $0, 0, \pm\omega i$ corresponding to the two degenerate directions resulting from scaling and rotating any relative equilibrium $\mathbf{x}$. The subspace W_2 of $\mathbb{C}^{4n}$ spanned by the four vectors

$$(\mathbf{u}, 0),\ (0, M\mathbf{u}),\ (\mathbf{v}, 0),\ (0, M\mathbf{v})$$

where $\mathbf{u} = (1, 0, 1, 0, \dots)$ and $\mathbf{v} = (0, 1, 0, 1, \dots)$ is also invariant under A with eigenvalues $\pm\omega i, \pm\omega i$ corresponding to the drift in the center of mass. The matrix A restricted to W_2 is not diagonalizable. In the classical sense then, any relative equilibrium is degenerate and not linearly stable.

We will follow Moeckel's approach in [10] and define linear stability by restricting A to the skew-orthogonal complement of these subspaces. See [10] for details.

Definition: A relative equilibrium $\mathbf{x}$ always has the eigenvalues $0, 0, \pm\omega i, \pm\omega i, \pm\omega i$. We will say that $\mathbf{x}$ is *nondegenerate* if the remaining $4n - 8$ eigenvalues are nonzero. It is *spectrally stable* if the eigenvalues are pure imaginary and is *linearly stable* if in addition, the restriction of the matrix A to the skew-orthogonal complement of $W_1 \cup W_2$ is diagonalizable.

2.3 The Characteristic Polynomial

Suppose that $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, and write $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^{2n}$. The eigenvector equation $A\mathbf{v} = \lambda\mathbf{v}$ then reduces to

$$\begin{aligned} \mathbf{v}_2 &= M(\lambda I - \omega K)\mathbf{v}_1 \\ B\mathbf{v}_1 &= 0 \end{aligned}$$

where

$$B = M^{-1}S + (\omega^2 - \lambda^2)I + 2\lambda\omega K. \tag{9}$$

Therefore, to obtain the eigenvalues of A, one need only take the determinant of B and find the roots in λ. In other words, $P(\lambda) = \det(B)$.

A closer examination of the structure of B provides some insight into $P(\lambda)$. The 2×2 blocks on the diagonal of B are given by

$$\begin{bmatrix} d_{i\,i} + \omega^2 - \lambda^2 & d_{i\,i+1} + 2\lambda\omega \\ d_{i+1\,i} - 2\lambda\omega & d_{i+1\,i+1} + \omega^2 - \lambda^2 \end{bmatrix} \tag{10}$$

where d_{ij} is the ij-th entry of $M^{-1}S$ (so i is odd in the above expression). Note that the only entries in B which contain λ are on the diagonal blocks. The determinant of matrix (10) is

$$\lambda^4 + (2\omega^2 - d_{i\,i} - d_{i+1\,i+1})\,\lambda^2 + (d_{i\,i} + \omega^2)(d_{i+1\,i+1} + \omega^2)$$

using the fact that $d_{i\,i+1} = d_{i+1\,i}$. We then can conclude that

$$P(\lambda) = \lambda^{4n} + (2n\,\omega^2 - \mathrm{tr}(M^{-1}S))\,\lambda^{4n-2} + \cdots$$

since the only terms which contribute to the coefficient of λ^{4n-2} are on one of the n diagonal blocks. Since $P(\lambda)$ is even, we now have a formula for the sum of the squares of the eigenvalues:

$$\frac{1}{2}\sum_{i=1}^{4n} \lambda_i^2 \;=\; \mathrm{tr}(M^{-1}S) - 2n\,\omega^2$$

where λ_i is an eigenvalue of A.

Direct computation reveals that

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix}$$

where S_{ij} is the 2×2 matrix given by

$$\begin{aligned} S_{ij} &= \frac{m_i m_j}{r_{ij}^3}\left[I - 3\mathbf{x}_{ij}\mathbf{x}_{ij}^{\mathrm{T}}\right] \quad \text{if} \quad i \neq j \\ S_{ii} &= -\sum_{i \neq j} S_{ij} \end{aligned}$$

and $\mathbf{x}_{ij} = \frac{\mathbf{x}_j - \mathbf{x}_i}{r_{ij}}$ (I is the 2×2 identity matrix, $\mathbf{x}_i$ is the position of the i-th body and r_{ij} is the distance between the i-th and j-th bodies). From this we calculate

$$\mathrm{tr}(M^{-1}S) \;=\; \sum_{i<j}^{n} \frac{m_i + m_j}{r_{ij}^3}\,.$$

Theorem 2.1 *Let $\mathbf{x}$ be a relative equilibrium with rotation speed ω given by formula (7). Then one-half of the sum of the squares of the eigenvalues of $\mathbf{x}$ is*

$$\sum_{i<j} \frac{m_i + m_j}{r_{ij}^3} \;-\; 2n\,\omega^2\,. \tag{11}$$

For example, consider the simple case $n = 2$. We know that any relative equilibrium has the eight eigenvalues $0, 0, \pm\omega i, \pm\omega i, \pm\omega i$. Since there are only eight eigenvalues when $n = 2$, one-half the sum of the squares of the eigenvalues is $-3\omega^2$. Using equations (4) and (5), any two-body relative equilibrium satisfies

$$\frac{m_2(-\frac{m_1}{m_2}\mathbf{x}_1 - \mathbf{x}_1)}{r_{12}^3} + \omega^2\,\mathbf{x}_1 = 0$$

or $\omega^2 = (m_1 + m_2)/r_{12}^3$. Formula (11) then yields $-3\omega^2$, as desired.

If a relative equilibrium $\mathbf{x}$ is to be spectrally stable, then the sum of the squares of the eigenvalues must be negative. Using this, in addition to the eight eigenvalues we already know, we have the following result:

Corollary 2.1 *A necessary condition for a relative equilibrium* $\mathbf{x}$ *to be spectrally stable is*

$$\sum_{i<j} \frac{m_i + m_j}{r_{ij}^3} \;<\; (2n-3)\,\frac{U(\mathbf{x})}{2I(\mathbf{x})}\,. \tag{12}$$

Our necessary condition for spectral stability is useful for a heuristic understanding of why many relative equilibria are unstable. If we fix the scaling of a relative equilibrium $\mathbf{x}$ so that $2I(\mathbf{x}) = 1$, then inequality (12) reduces to

$$\sum_{i<j} \frac{m_i + m_j}{r_{ij}^3} \;<\; (2n-3)\sum_{i<j} \frac{m_i m_j}{r_{ij}}. \tag{13}$$

As n gets larger, the bodies on the mass ellipsoid determined by $2I(\mathbf{x}) = 1$ must be getting closer and closer. This means that the $(m_i + m_j)/r_{ij}^3$ terms begin to dominate the $m_i m_j / r_{ij}$ terms, and consequently, the left-hand side of inequality (13) becomes larger than the right-hand side, and the configuration is unstable. We use this line of reasoning in Section 4 to show that the equal mass relative equilibria are unstable for n sufficiently large.

3 Applications

Before considering the equal mass case, we demonstrate the usefulness (and simplicity) of inequality (12) with two well-known relative equilibria.

3.1 Regular n-gon

Consider the relative equilibrium consisting of n equal masses at the vertices of a regular n-gon. Moeckel shows that this configuration is not spectrally stable for any n in [9]. Here we show that our condition gives a simple proof of the instability of the regular n-gon for $n \geq 7$.

We set $m_j = 1$, for $j \in \{1, \ldots, n\}$ and let the position of the jth body be given by $\mathbf{x}_j = (\cos\theta_j, \sin\theta_j)$ where $\theta_j = 2\pi j/n$. Recall that r_{ij} represents the distance between the i-th and j-th bodies. We will make use of the formula

$$r_{nj} = 2\sin(\pi j/n). \tag{14}$$

Note that by symmetry, $r_{i\,i+j} = r_{nj}$ with the indices taken mod n. Since $I = n/2$ for this configuration, using $\omega^2 = U/(2I)$, inequality (12) becomes

$$\sum_{i<j} \frac{1}{r_{ij}^3} \;<\; (1 - \frac{3}{2n})\sum_{i<j} \frac{1}{r_{ij}}.$$

Using the symmetry of the configuration, this reduces to

$$\sum_{j=1}^{n-1} \frac{1}{r_{nj}^3} \;<\; (1-\frac{3}{2n})\sum_{j=1}^{n-1} \frac{1}{r_{nj}}.$$

Then, using formula (14), a necessary condition for the regular n-gon to be linearly stable is

$$\sum_{j=1}^{n-1} \csc^3 \frac{\pi j}{n} \;<\; \sum_{j=1}^{n-1} (4-\frac{6}{n}) \csc \frac{\pi j}{n}. \tag{15}$$

Lemma 3.1 *Inequality (15) is only satisfied for $3 \le n \le 6$. Consequently, the regular n-gon is not spectrally stable for $n \ge 7$.*

Proof: We can numerically evaluate inequality (15) for $3 \le n \le 16$. For $n \ge 17$, we make the rough estimates

$$\sum_{j=1}^{n-1} \csc^3 \frac{\pi j}{n} \;>\; 2\csc^3 \frac{\pi}{n} \qquad \text{and} \qquad \sum_{j=1}^{n-1} \csc \frac{\pi j}{n} \;<\; (n-1)\csc \frac{\pi}{n},$$

which then yields

$$\sum_{j=1}^{n-1} \csc^3 \frac{\pi j}{n} - (4-\frac{6}{n}) \csc \frac{\pi j}{n} \;>\; \csc \frac{\pi}{n}(2\csc^2 \frac{\pi}{n} - (4-\frac{6}{n})(n-1)).$$

Using $\csc x > 1/x$, we have

$$2\csc^2 \frac{\pi}{n} - (4-\frac{6}{n})(n-1) \;>\; \frac{2n^2}{\pi^2} - 4n + 10 - \frac{6}{n}$$

which is positive for $n \ge 17$. This completes the proof.

The regular n-gon for $3 \le n \le 6$ is thus an example of a relative equilibrium where the sum of the eigenvalues squared is negative (because inequality (12) is satisfied) yet the configuration is known to be unstable. Not surprisingly, this means that our condition (12) is necessary for spectral stability but certainly not sufficient.

3.2 $1+n$-gon

By adding a body of mass m at the center of the regular n-gon configuration, we obtain a family of relative equilibria for the $(n+1)$-body problem, with m serving as a parameter. The British scientist James Clerk Maxwell first studied this family in his paper on the stability of Saturn's rings, written for the 1856 Adams Prize Essay [5, 6]. He concluded that the ring was stable, provided the central mass was sufficiently large. As a slight correction to this, Moeckel proves in [9] and in [10], that the $1+n$-gon is linearly stable for sufficiently large m, only when $n \ge 7$. Here we show that inequality (12) is satisfied for m sufficiently large.

We will keep the same set up as we used in the regular n-gon. In addition, let $m_{n+1} = m$ and $\mathbf{x}_{n+1} = (0,0)$. Note that $I = n/2$ for this configuration also. Inequality (12) reduces to

$$\sum_{i<j} \frac{2}{r_{ij}^3} + n(1+m) \;<\; (2-\frac{1}{n})\left[\sum_{i<j}^{n} \frac{1}{r_{ij}} + nm\right]$$

or

$$\sum_{i<j} \frac{1}{r_{ij}^3} \;<\; (1-\frac{1}{2n})\sum_{i<j}^{n} \frac{1}{r_{ij}} + \frac{m}{2}(n-1) - \frac{n}{2}.$$

[Note that we replace n with $n+1$ in inequality (12).] Again, using the symmetry of the n-gon, this reduces to

$$\frac{n}{2}\sum_{j=1}^{n-1} \frac{1}{r_{nj}^3} \;<\; (1-\frac{1}{2n})\,\frac{n}{2}\sum_{j=1}^{n-1} \frac{1}{r_{ij}} + \frac{m}{2}(n-1) - \frac{n}{2}$$

which is satisfied when

$$m \;>\; \frac{n}{n-1}\left(\sum_{j=1}^{n-1} (\frac{1}{r_{nj}^3} - \frac{2n-1}{2n}\cdot\frac{1}{r_{nj}}) + 1\right). \tag{16}$$

While this does not reveal the instability of the $1+n$-gon for $3 \leq n \leq 6$, it does provide a lower bound for the size of the central mass in order to achieve stability for $n \geq 7$. In fact, a little calculation shows that the term on the right-hand side of (16) is asymptotic to $\tau\, n^3$ where

$$\tau \;=\; \frac{1}{4\pi^3}\cdot\sum_{j=1}^{\infty} \frac{1}{j^3} \;\approx\; 0.009692.$$

(By asymptotic we mean that the limit as n goes to infinity of the ratio between n^3 and the right-hand side of (16) is τ.) The actual asymptotic value of the central mass for which the $1+n$-gon becomes linearly stable, as derived in [11], is approximately $.435n^3$ (a similar result is obtained in [2]). Thus, our simple inequality reveals the correct order of the central mass necessary for stability.

4 Relative Equilibria with Equal Masses

Theorem 4.1 *Any relative equilibrium of n equal masses is not spectrally stable for $n \geq 24,306$.*

We show that in the equal mass case, inequality (12) is not satisfied for $n \geq 24,306$. This argument holds for all equal mass relative equilibria. Indeed, for large n, there are a tremendous number of relative equilibria [7]. However, we can estimate inequality (12) for all of them without knowing their exact positions.

We first provide the motivation and details behind the proof. We perform several calculations which may seem careless (ignoring greatest integer signs for example) but are intended only to obtain expressions explaining where the actual parameter values come from for the proof at the end of this section.

The goal is to show that

$$\sum_{i<j} \frac{m_i + m_j}{r_{ij}^3} \;-\; 2n\,\omega^2 \;>\; 0$$

(we choose $2n$ as opposed to $2n-3$ to simplify the calculations). Setting $m_i = 1$ for each i and recalling formula (7) for ω^2, this reduces to showing

$$\sum_{i<j} \frac{1}{r_{ij}^3} \;-\; \frac{n}{2I(\mathbf{x})}\sum_{i<j} \frac{1}{r_{ij}} \;>\; 0.$$

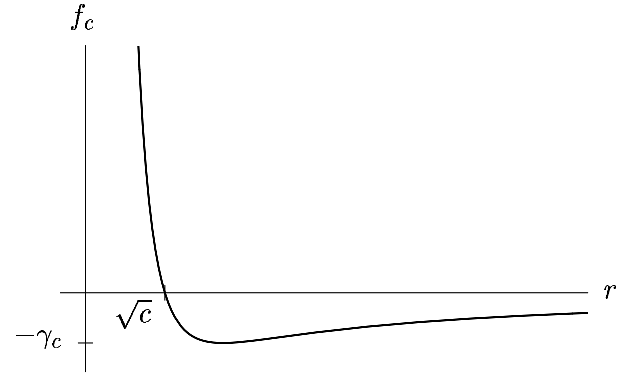

Figure 1: Graph of the function $f_c(r) = 1/r^3 - 1/(cr)$.

This motivates fixing $2I = c\,n$ or

$$\sum_{i=1}^{n} \|\mathbf{x}_i\|^2 \;=\; c\,n,$$

with the constant $c > 0$ to be chosen later. It then suffices to show that

$$\sum_{i<j} \frac{1}{r_{ij}^3} \;-\; \frac{1}{c} \cdot \frac{1}{r_{ij}} \;>\; 0. \tag{17}$$

Fixing $2I = c\,n$ sets a bound on the distances bodies can be from the origin. We will assume that $c < 1$. Then we are guaranteed at least $[n(1-c)]$ bodies inside the unit disk D, where $[\;\;]$ denotes the greatest integer. Otherwise, we would have $n - [n(1-c)]$ bodies outside or on D, and the value of $2I$ would be larger than cn, a contradiction. As n gets larger, although the bound grows in size, more and more bodies must be inside D and in particular, closer to each other so that the $1/r_{ij}^3$ terms start to dominate the $1/r_{ij}$ terms.

Let $f_c(r) = 1/r^3 - 1/(cr)$ for $r > 0$. It is easy to check that $f_c(r) > 0$ for $0 < r < \sqrt{c}$, $f_c(r) \leq 0$ for $r \geq \sqrt{c}$, and that f_c has a global minimum of $-\gamma_c = -2(3c)^{-3/2}$ (see Figure 1). We wish to cover D with regions such that any two bodies in a region are within ϵ of each other, where ϵ is a small parameter with $\epsilon < \sqrt{c}$. Let K_ϵ be the number of regions of diameter ϵ needed to cover D.

Next, let

$$N_{c,\epsilon} = \left[\frac{[n(1-c)]}{K_\epsilon} \right]$$

be the approximate number of bodies in each ϵ-region (if they were to be evenly distributed). If ϵ is chosen small enough, and $N_{c,\epsilon}$ is large enough, then we will have lots of bodies close to each other helping us verify inequality (17). In fact, we claim that

$$\sum_{i<j} \frac{1}{r_{ij}^3} \;-\; \frac{1}{c} \cdot \frac{1}{r_{ij}} \;>\; f_c(\epsilon) \cdot K_\epsilon \binom{N_{c,\epsilon}}{2} \;-\; \gamma_c \binom{n}{2}. \tag{18}$$

To justify this claim, we make the following argument. Suppose that ϕ represents a distribution of n balls into K numbered baskets ($n > 2K$) and let Φ represent the set of all possible distributions. For a given distribution ϕ, let

$$R_\phi = \sum_{i=1}^{K} \binom{n_i}{2}$$

where n_i is the number of balls in the i-th basket. We have the following fact:

Lemma 4.2 *Let $N = [n/K]$. Then $K \binom{N}{2} \leq \min_{\phi \in \Phi} R_\phi$*

Proof: The proof is a counting argument. Putting exactly N balls in each of the K baskets (forgetting any leftovers) yields $K \binom{N}{2}$ as a value for R_θ. Also, if n_i and n_j are positive integers such that $n_i \geq n_j$, we have

$$\binom{n_i}{2} + \binom{n_j}{2} < \binom{n_i+1}{2} + \binom{n_j-1}{2}.$$

This inequality means that if we move a ball from one basket into another basket with fewer balls, the value R_ϕ decreases. The result now follows.

This lemma allows us to conclude that a lower bound on the number of "small" distances is obtained through an equal distribution of bodies in the K_ϵ ϵ-regions. For every pair of bodies in an ϵ-region, we know that at least $f_c(\epsilon)$ is contributed positively to the sum on the left-hand side of inequality (18).

The worst or most negative a term in inequality (18) can be is $-\gamma_c$. We seek an upper bound on the number of "big" distances which contribute negatively to the sum on the left-hand side of inequality (18). In other words, we would like to find as many bodies as possible within $\sqrt{c}$ of each other. While these bodies may not contribute as much as $f_c(\epsilon)$ to the sum, at least they are a positive contribution, and therefore should not be counted with the γ_c terms.

Following the arguments presented above, we let D_α be the circle of radius $\sqrt{\alpha}$ centered at the origin and $K_{\sqrt{c},\alpha}$ be the number of regions of diameter $\sqrt{c}$ needed to cover D_α (α is a parameter to be determined). By fixing $2I = cn$, we are guaranteed $[n(1 - c/\alpha)]$ bodies inside D_α. For this to make sense, we will assume that $\alpha > c$. We let

$$N_{c,\alpha} = \left[\frac{[n(1 - c/\alpha)]}{K_{\sqrt{c},\alpha}} \right]$$

be the approximate number of bodies in each $\sqrt{c}$-region if they were to be evenly distributed. Then Lemma 4.2 assures us that we have at least

$$P_{c,\alpha} = K_{\sqrt{c},\alpha} \binom{N_{c,\alpha}}{2}$$

"big" distances which contribute positively to the sum on the left-hand side of inequality (18).

We wish to find a value of α that maximizes $P_{c,\alpha}$. Since D_α has area $\pi\alpha$ and a circle of diameter $\sqrt{c}$ has area $\pi c/4$, we will assume that $K_{\sqrt{c},\alpha} = \hat{h}(4\alpha/c)$ where $\hat{h}$ is some proportionality constant between 1 and 2. We then compute

$$P_{c,\alpha} = \frac{n(1 - c/\alpha)}{2} \cdot \left(\frac{n(1 - c/\alpha)}{\hat{h}(4\alpha/c)} - 1 \right)$$

(we drop the greatest integer for the moment to simplify the calculations). This motivates setting $x = c/\alpha$ and considering the function

$$g_n(x) = \frac{n(1-x)}{2} \cdot \left(\frac{n\,x(1-x)}{4\hat{h}} - 1 \right) \quad \text{for } 0 \leq x \leq 1.$$

The function g_n has a maximum on the interval $(0, 1)$ at the value

$$x_n = \frac{2}{3} - \frac{1}{3}\sqrt{1 - 12\hat{h}/n}$$

which we will approximate by $1/3$ since we expect n to be large. Thus, setting $\alpha = 3c$, we obtain an approximate maximum for $P_{c,\alpha}$ by

$$P_{c,3c} \;=\; g_n(1/3) \;=\; \frac{n^2}{54\hat{h}} - \frac{n}{3}.$$

Therefore, we are guaranteed at least $P_{c,3c}$ mutual distances which are less than $\sqrt{c}$. We now have a slightly improved estimate from (18):

$$\begin{aligned}
\sum_{i<j} \frac{1}{r_{ij}^3} - \frac{1}{c}\cdot\frac{1}{r_{ij}} &> f_c(\epsilon)\cdot K_\epsilon \binom{N_{c,\epsilon}}{2} - \gamma_c\left(\binom{n}{2} - P_{c,3c}\right) \\
&= f_c(\epsilon)\cdot K_\epsilon \binom{N_{c,\epsilon}}{2} - \gamma_c\left((\frac{1}{2} - \frac{1}{54\hat{h}})n^2 - \frac{n}{6}\right) \\
&= A_{c,\epsilon}\, n^2 + B_{c,\epsilon}\, n
\end{aligned}$$

where

$$\begin{aligned}
A_{c,\epsilon} &= \frac{9\sqrt{c}\,(1-c)^2(c-\epsilon^2) - 8\sqrt{3}\,h\epsilon\,(1 - 1/27\hat{h})}{72h\,\epsilon\,c^{3/2}}, \\
B_{c,\epsilon} &= \frac{2\sqrt{3}\,\epsilon^3 - 27\sqrt{c}\,(1-c)(c-\epsilon^2)}{54\epsilon^3 c^{3/2}}.
\end{aligned}$$

Here again, we have disregarded the greatest integer for $N_{c,\epsilon}$ and assumed that $K_\epsilon = h(4/\epsilon^2)$ with h some proportionality constant between 1 and 2.

Note that $A_{c,\epsilon}$ is positive for ϵ sufficiently small. This means that $A_{c,\epsilon}\, n^2 + B_{c,\epsilon}\, n$ is simply a parabola in n opening upwards. Thus, for n sufficiently large, inequality (17) will hold.

A simple estimate shows that $A_{c,\epsilon} > 0$ implies $B_{c,\epsilon} < 0$. Indeed, since ϵ and c are chosen so that $\epsilon < \sqrt{c}$ and $c < 1$, we have

$$\begin{aligned}
27\sqrt{c}\,(1-c)(c-\epsilon^2) &> 27\sqrt{c}\,(1-c)^2(c-\epsilon^2) \\
&> 24\sqrt{3}\,h\epsilon(1 - 1/27\hat{h}) \\
&> 24\sqrt{3}\,\epsilon^3(26/27) \\
&> 2\sqrt{3}\,\epsilon^3.
\end{aligned}$$

Here we have used the assumption that $\hat{h}$ and h are larger than one. It follows that when $A_{c,\epsilon} > 0$, the root $-B_{c,\epsilon}/A_{c,\epsilon}$ is positive. We need to choose c and ϵ such that:

1. $A_{c,\epsilon} > 0$, and

2. $-B_{c,\epsilon}/A_{c,\epsilon}$ is as small as possible.

Once this is achieved, we will have shown that inequality (17) is satisfied for $n > -B_{c,\epsilon}/A_{c,\epsilon}$.

Before we can find optimal values of c and ϵ, we need to decide upon values for $\hat{h}$ and h. Recall that we "covered" D_α with $K_{\sqrt{c},\alpha} = \hat{h}(4\alpha/c)$ regions, such that any two bodies in a region were

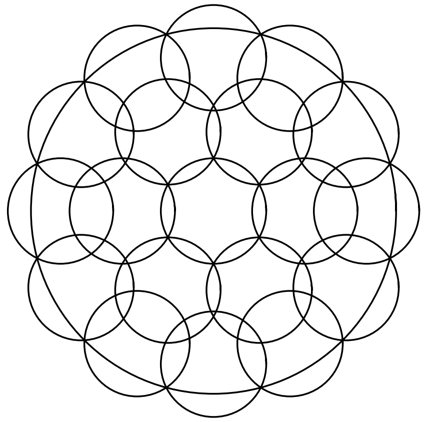

Figure 2: Covering a circle of radius $\sqrt{3c}$ with 19 circles of radius $\sqrt{c}/2$.

at most $\sqrt{c}$ apart. We also decided to set $\alpha = 3c$. A value of $\hat{h} = 1$, implies that we could cover D_{3c} with 12 circles of radius $\sqrt{c}/2$. But this is obviously impossible. We can cover a circle of radius $\sqrt{3c}$ with 19 circles of radius $\sqrt{c}/2$ (see Figure 2). This gives us a proportionality constant of $\hat{h} = 19/12 = 1.58\bar{3}$.

Finding a value for h is slightly different. We want to cover the unit disc D with regions of diameter ϵ. We would need to use at least $4/\epsilon^2$ circles. To make sure that we cover D completely, we will use hexagons instead of circles. A regular hexagon with side length $\epsilon/2$ has area $3\sqrt{3}\,\epsilon^2/8$. Thus, we would need at least $8\pi/(3\sqrt{3}\,\epsilon^2)$ hexagons to cover D. For good measure, we also include enough hexagons to cover the circumference of D. This gives us a value of

$$\begin{aligned} K_\epsilon &= \frac{8\pi}{3\sqrt{3}\,\epsilon^2} + \frac{4\pi}{\epsilon} \\ &= \frac{4}{\epsilon^2}\left(\frac{8\pi}{12\sqrt{3}} + \pi\epsilon\right) \end{aligned}$$

which means $h \approx 1.2092 + \pi\epsilon$.

Plugging these values of $\hat{h}$ and h into $A_{c,\epsilon}$ and then using a computer to find the minimum of $-B_{c,\epsilon}/A_{c,\epsilon}$ yields the values $\epsilon = 0.029255$, $c = 0.370093$ and $[-B_{c,\epsilon}/A_{c,\epsilon}] = 26,975$.

Proof of Theorem 4.1:
We set $\epsilon = 0.029255$, $c = 0.370093$ and $\alpha = 3c$. We cover D with $K_\epsilon = [4h/\epsilon^2] + 1 = 6,081$ regions of diameter ϵ and cover D_α with 19 regions of diameter $\sqrt{c}$. For any $n \geq 24,306$, define the natural numbers q and $r < K_\epsilon$ by

$$[n(1-c)] = q \cdot K_\epsilon + r$$

and similarly, $\hat{q}$ and $\hat{r} < 19$ by

$$[n(2/3)] = \hat{q} \cdot 19 + \hat{r}.$$

Then, for any relative equilibrium of n equal masses, we have

$$\begin{aligned} \sum_{i<j} \frac{1}{r_{ij}^3} - \frac{1}{c} \cdot \frac{1}{r_{ij}} \quad > \quad & f_c(\epsilon)\left((K_\epsilon - r)\binom{q}{2} + r\binom{q+1}{2}\right) - \\ & \gamma_c\left(\binom{n}{2} - (19-\hat{r})\binom{\hat{q}}{2} - \hat{r}\binom{\hat{q}+1}{2}\right). \end{aligned}$$

Here we have used a variation of Lemma 4.2 to obtain a minimum distribution with q bodies in some regions and $q+1$ bodies in the rest. A simple computer program was written to show that the right-hand side of the previous expression is positive for $24,306 \leq n \leq 26,975$. For the case $n \geq 26,976$, we have

$$\begin{aligned}
\sum_{i<j} \frac{1}{r_{ij}^3} - \frac{1}{c} \cdot \frac{1}{r_{ij}} \quad &> \quad f_c(\epsilon) \left((K_\epsilon - r) \begin{pmatrix} q \\ 2 \end{pmatrix} + r \begin{pmatrix} q+1 \\ 2 \end{pmatrix} \right) - \\
&\quad \gamma_c \left(\begin{pmatrix} n \\ 2 \end{pmatrix} - (19 - \hat{r}) \begin{pmatrix} \hat{q} \\ 2 \end{pmatrix} - \hat{r} \begin{pmatrix} \hat{q}+1 \\ 2 \end{pmatrix} \right) \\
&> \quad f_c(\epsilon) \left(K_\epsilon \cdot \frac{n(1-c)}{2K_\epsilon} \left(\frac{n(1-c)}{K_\epsilon} - 1 \right) \right) - \\
&\quad \gamma_c \left(\begin{pmatrix} n \\ 2 \end{pmatrix} - 19 \cdot \frac{n(2/3)}{2 \cdot 19} \left(\frac{n(2/3)}{19} - 1 \right) \right) \\
&> \quad 0.465215916\, n^2 - 12,549.63\, n
\end{aligned}$$

which is positive for $n \geq 26,976$. We have shown that inequality (12) does not hold for any relative equilibrium of n equal masses for $n \geq 24,306$. This completes the proof.

Remarks:

1. Using the same arguments as above, it is possible to show that any *collinear* equal mass relative equilibrium is spectrally unstable for $n \geq 22$. (It is known that all collinear relative equilibria are spectrally unstable [9].)

2. The optimal result would be that for **any** n, the equal mass relative equilibria are all spectrally unstable. The main purpose of our argument here is to demonstrate how inequality (12) can be of use for n sufficiently large. We do not expect inequality (12) to be helpful for small n because there are several relative equilibria which satisfy it but are not stable (for example, the regular n-gon of Section 3.1). While it may be possible to improve the above estimates, a different method is required to prove that all equal mass relative equilibria are unstable.

3. Attempts were made to improve the estimates above by considering more "small" distances (for example, bodies in adjacent regions would be within 2ϵ of each other). However, this is complicated because the equivalent to Lemma 4.2 does not exist. It is not clear what the minimum distribution is when considering more than just distances within regions.

4. Another approach to improving the estimate is to replace the unit disk D with a disk of arbitrary radius $\sqrt{\delta}$ and then continue as above, searching for the best possible values of the parameters ϵ, c, and δ. However, the introduction of δ turns out to be superfluous as this method reduces to searching for the best possible values of the parameters $\epsilon/\sqrt{\delta}$ and c/δ.

Acknowledgments: I would like to thank Glen Hall for diligently checking this work and for his many suggestions regarding its presentation.

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