Chaos and Fractals in Music

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Chaos Theory

The theory of *Dynamical Systems* (more popularly known as Chaos Theory) focuses on the behavior occurring in a system under iteration. Graphical, numerical and analytic approaches are all important. Computers helped reinvigorate the field.

Example: $f(x) = x^2$. What are the fates of different orbits under iteration of f?

$$x_{n+1} = f(x_n)$$

$x_0 = 2$ is called an initial seed.

$x_1 = f(x_0) = f(2) = 4$ is the next iterate in the orbit.

$x_2 = f(x_1) = f(4) = 16$ and $x_3 = f(x_2) = f(16) = 256$ are the next two iterates.

The orbit of 2 is then $2, 4, 16, 256, \ldots$. It is heading off to ∞.

$f(x) = x^2$ (cont.)

$x_0 = 1$ has an orbit of

$$1, 1, 1, 1, 1, \ldots$$

We call $x_0 = 1$ a fixed point because it is fixed under iteration of f.

What's another fixed point of f?

Answer: $x_0 = 0$. The orbit of $x_0 = 0$ is $0, 0, 0, 0, \ldots$

These are the only two fixed points since solving the equation $f(x) = x$ is equivalent to solving $x^2 = x$.

$$x^2 - x = 0 \quad \text{or} \quad x(x - 1) = 0.$$

$f(x) = x^2$ (cont.)

$x_0 = -1$ has an orbit of $\quad -1, 1, 1, 1, 1, \ldots$
We call $x_0 = -1$ an eventually fixed point.

$x_0 = 1/2$ has an orbit of

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$$

which converges (limits) to 0.

Any x_0 satisfying $-1 < x_0 < 1$ will have an orbit that limits on the fixed point 0. Thus, we call $x_0 = 0$ an attracting fixed point.

If $x_0 > 1$ or $x_0 < -1$, the orbit approaches ∞. Since orbits starting close to $x_0 = 1$ move away under iteration, we call $x_0 = 1$ a repelling fixed point.

$$f(x) = x^2 - 1$$

Consider the dynamical system determined by $f(x) = x^2 - 1$.

What is the orbit of 0? $0, -1, 0, -1, 0 - 1, \ldots$

This orbit is called a period two cycle.

What about fixed points? Solving $f(x) = x$ yields the equation

$$x^2 - 1 = x \quad \text{or} \quad x^2 - x - 1 = 0.$$

Solution: $\dfrac{1 \pm \sqrt{5}}{2}$, the Golden ratio!

Goals of Dynamical Systems

- For a particular system, study the fate of all orbits under iteration. Find the fixed points, period two points, attractors and repellors, orbits that head off to ∞, chaotic orbits (unpredictable), etc.

- What is the underlying structure of the system? What does the set of all periodic points look like? Is it simple or complicated? What does the set of points whose orbits remain bounded (not heading off to ∞) look like?

- Look at systems with a parameter. For example, $Q_c(x) = x^2 + c$, where c can vary. What type of structural changes occur as c is varied? Where are the bifurcations? How do we characterize the dynamics for different c values?

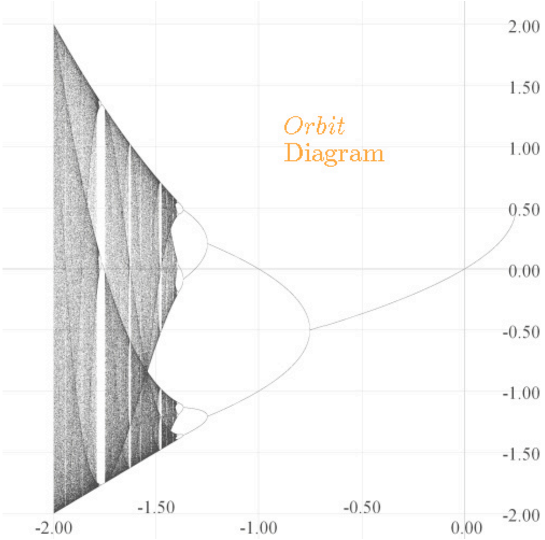


Figure: The orbit diagram for the quadratic family $Q_c(x) = x^2 + c$. For each c-value, the orbit of $x_0 = 0$ is computed up to 2200 iterations. The first 2000 iterations are discarded and the remaining 200 are shown.

Fractals

Figure: Fractals at the Museum of Science (Montserrat 2010-11)

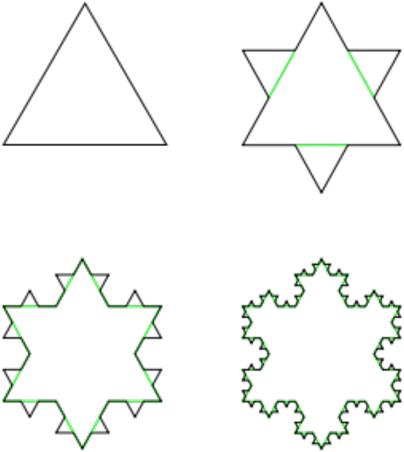


Figure: The first four iterations of the Koch snowflake curve (Helge von Koch, 1904), one of the earliest fractals.

Perimeter of the Koch Snowflake

Suppose each side of the equilateral triangle has a length of 1. At the opening stage, the perimeter is therefore 3.

At the next iterate, each "side" now contains four segments that are $1/3$ the original length. Each side has a perimeter of $4/3$ giving a total perimeter of $3 \cdot 4/3 = 4$.

At the next iterate, each new "side" now contains four segments that are $1/9$ the original length, for a total of $4/9$. But there are 12 such "sides" giving a total perimeter of $12 \cdot 4/9 = 16/3$.

The perimeter is growing!

$$3, 4, \frac{16}{3}, \frac{64}{9}, \ldots$$

This is a geometric sequence with a ratio $r = 4/3 > 1$. This goes to ∞. We have a curve enclosing finite area with infinite length — a monster!

The Butterfly Effect

One of the hallmarks of a chaotic dynamical system is sensitive dependence on initial conditions, more commonly known as the Butterfly Effect. The idea is that a small change in initial conditions can lead to a large change in the behavior of a system.

“Does the flap of a butterfly’s wings in Brazil set off a tornado in Texas?"

Term comes from Ed Lorenz, an MIT meteorologist who accidentally discovered it while trying to model the weather in 1961. Lorenz truncated his data in one run-through, entering 0.506 instead of 0.506127. To his great surprise, he found that the original results were vastly altered by this minor change in one variable. Where it was sunny, there was now rain; a windy day was now calm, etc.

The butterfly effect is the reason that weather prediction is so difficult.

The Butterfly Effect in Music

Examples of the butterfly effect in music:

- Steve Reich — small changes in rhythmic structure (e.g., slight phase shift) lead to big changes in the music (e.g., *Clapping Music*, *Violin Phase*, *Six Marimbas*, etc.)

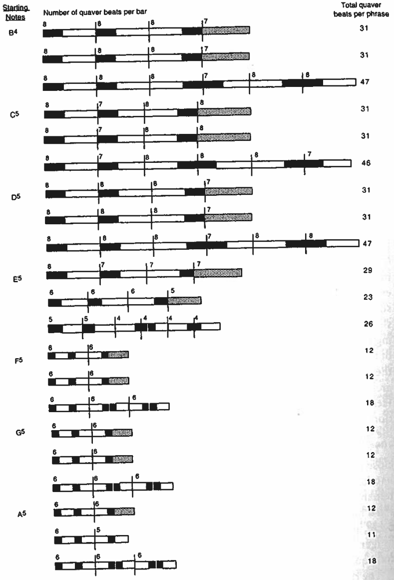
- György Ligeti (1923-2006). Hungarian composer who used mathematical ideas in many of his compositions. Popularly known for the music in the Stanley Kubric films *2001: A Space Odyssey* and *The Shining*.

  A musical depiction of the butterfly effect occurs in his piece *Désordre* (“Disorder," 1985) which is the first in a set of 18 technically challenging piano pieces titled *Études*. The right hand plays solely the white keys (C major) while the left hand only the black keys (a pentatonic scale).

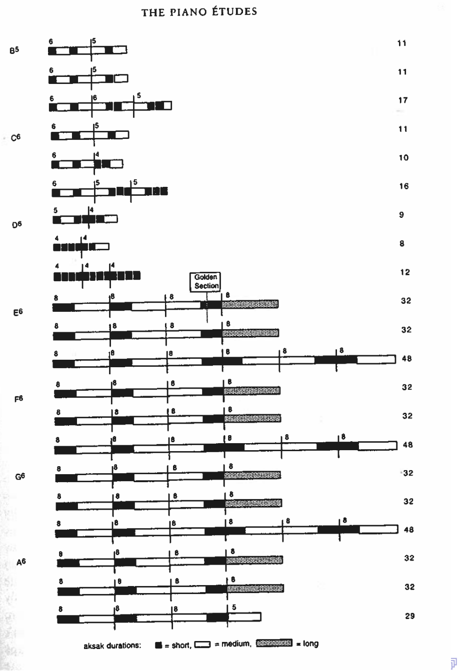
Désordre (1985)

- Each hand opens with identical 8-beat rhythmic patterns $(3+5)$. In the fourth measure, the right hand drops a beat, playing a 7-beat pattern rather than an 8-beat one, but continues the 8-beat pattern for the next three measures. This small change starts to cause a big shift, audible for the listener due to the shifting accents in each hand.
- In the eighth measure, the right hand drops another beat, playing 7 instead of 8. Now, the left hand is two beats ahead instead of one. Again, the right hand only drops a beat in this one measure.
- The "iterative" process of dropping a single beat continues, as the right hand drops a beat approximately every four measures so that the two hands become completely out of synch, and the butterfly effect is realized. Each successive deletion of a beat in the right hand is denoted by a vertical dashed line in the score.


ÉTUDE 1: „DÉSORDRE"
Dédiée à Pierre Boulez
György Ligeti
1985
Molto vivace, Vigoroso, molto ritmico
sempre simile
Piano
sempre legatissimo possibile
Stets sehr sparsamer Gebrauch des Pedals / Utilisez la pédale très discrètement (pendant toute la pièce)



Ex. 24 *Désordre*, phrase and rhythmic structure of the right hand


THE PIANO ÉTUDES
B5
C6
D6
E6
F6
G6
A6
Golden Section
11
11
17
11
10
16
9
8
12
32
32
48
32
32
48
32
32
48
32
32
29
aksak durations: = short, = medium, = long

Désordre (cont.)

- The right-hand part can be subdivided into sections whose starting notes slowly ascend up the C major scale. The piece reaches a dynamic and rhythmic climax a few bars away from the golden section, after which the two hands play in unison 8-beat rhythmic phrases again.
- This time the right hand maintains the same rhythmic phrasing while the left hand gradually adds a beat, going from an 8-beat phrase to a 9-beat phrase on occasion.
- As in the opening half, the “iterative” process of adding a single beat continues, but this time a bit quicker, as the left hand adds a beat approximately every three (not four) measures so that the two hands become out of synch a bit faster.


8va
Dynamische Balance: die rechte Hand etwas kräftiger spielen als die linke, so dass die akzentuierten Akkorde in beiden Händen gleich laut klingen (bis Ende des Stückes)
allmählich etwas mehr Pedal (doch stets sparsam)
Allmähliches crescendo (bis Ende): die Akzente werden allmählich ff, dann fff (mit stets stärkerer rechten Hand), die Achtel-Figuren allmählich mp, dann mf.


8va
8va
8va
8va
8va