Seminar in Mathematics and Climate

Homework Assignment #6

Due Fri., April 6, 5:00 pm

You should write up solutions neatly to all problems, making sure to show your work. A nonempty subset will be graded. You are encouraged to work on these problems with other classmates, and it is ok to use internet sources for help if it's absolutely necessary (with proper citation); however, the solutions you turn in should be your own work and written in your own words.

Note: For this assignment, you are allowed to work in groups of up to four people. Please turn in one assignment per group.

1. Consider the one-parameter family of linear systems $\dot{X} = AX$ and $\dot{Y} = BY$, where

$$A = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b & b \\ 1 & 0 \end{bmatrix}.$$

Thus a is the parameter for the first system and b is the parameter in the second system. For each system, answer the following questions.

(a) Find the trace T and determinant D in terms of the given parameter. Then sketch the curve that is traced out in the TD-plane as the parameter is varied.

(b) Find the bifurcation values, that is, find the values of the parameter where the type of equilibrium point changes.

(c) In a paragraph, describe the different types of equilibrium points possible as the parameter varies. Be sure to focus on the behavior before, at, and after each bifurcation.

2. Consider the following nonlinear system of differential equations describing the motion of a damped pendulum:

$$\begin{aligned} \frac{d\theta}{dt} &= v \\ \frac{dv}{dt} &= -v - 2\sin\theta. \end{aligned}$$

Here, θ measures the angle between the vertical axis and the pendulum arm, so $-\pi \leq \theta \leq \pi$, and v is the angular velocity (see Figure 1).

(a) Find the three equilibrium points for the system and explain the meaning of each point in terms of the pendulum.

(b) Linearize the system about each equilibrium point and then classify each equilibrium (sink, source, saddle, etc.). Interpret the stability of each equilibrium point in terms of the motion of the pendulum.

(c) Sketch the phase plane portrait near each equilibrium point.

(d) Using your answer to part **(c)**, sketch the complete phase portrait in the θv-plane. Feel free to use technology to help with your sketch (e.g., the dfield Java app).

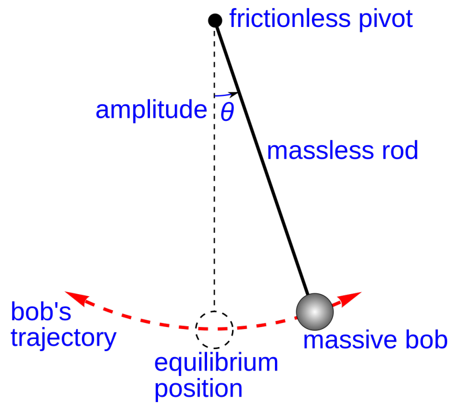

Figure 1: The swinging pendulum.

3. Recall Stommel's two-box model for ocean circulation:

$$\begin{aligned} x' &= \delta(1-x) - |f|x \\ y' &= 1 - y - |f|y \\ \lambda f &= -y + Rx. \end{aligned} \tag{1}$$

The Jacobian matrix for this system depends on the sign of f. We obtain the matrix

$$\begin{bmatrix} -\delta - |f| - \frac{R}{\lambda}x & \frac{1}{\lambda}x \\ -\frac{R}{\lambda}y & -1 - |f| + \frac{1}{\lambda}y \end{bmatrix} \text{ for } f > 0, \text{ or } \begin{bmatrix} -\delta - |f| + \frac{R}{\lambda}x & -\frac{1}{\lambda}x \\ \frac{R}{\lambda}y & -1 - |f| - \frac{1}{\lambda}y \end{bmatrix} \text{ for } f < 0.$$

Define the function $G(f; R, \delta) = \dfrac{R\delta}{\delta + |f|} - \dfrac{1}{1 + |f|}$. Recall that system (1) has an equilibrium point at

$$(x, y) = \left(\frac{\delta}{\delta + |f|}, \frac{1}{1 + |f|} \right) \tag{2}$$

provided that f satisfies the equation $G(f) = \lambda f$.

(a) For either case $f > 0$ or $f < 0$, show that the trace T of the Jacobian is $T = -(1+\delta+3|f|)$. What does this immediately imply about the stability type of any equilibrium point? What type of solution is impossible to have?

(b) Stommel takes the values $R = 2$, $\delta = 1/6$, and $\lambda = 1/5$ in his paper. He finds three equilibria corresponding to the values $f_1 \approx -1.06791$, $f_2 \approx -0.30703$, and $f_3 \approx 0.21909$. Using the correct Jacobian along with equation (2), check that these correspond to a real sink, saddle, and spiral sink, respectively. State the trace T and determinant D in each case.

(c) Let $R = 3/2$, $\delta = 1/6$, and $\lambda = 1/2$. Find the equilibrium points and determine the stability type of each one. Give the values of f, T, and D at each point (5 decimal places).