Chapters 3 and 6: Oceans and Climate

Gareth E. Roberts

Department of Mathematics and Computer Science
College of the Holy Cross
Worcester, MA, USA

Seminar in Mathematics and Climate
MATH 392-01 Spring 2018
February 27, March 1, 15, and 20, 2018

Importance of the Oceans

- Oceans play a critical role in the Earth's climate system. They cover around 71% of the surface area of the planet.
- Two important functions:
  1. Heat transport (e.g., the $C(T - \overline{T})$ term in Budyko's EBM)
  2. Absorb large amounts of CO_2 from atmosphere
- CO_2 in ocean consumed by tiny single-cell organisms (phytoplankton) through photosynthesis. They are eventually food for larger species or sink to the bottom of the ocean once the plankton dies.

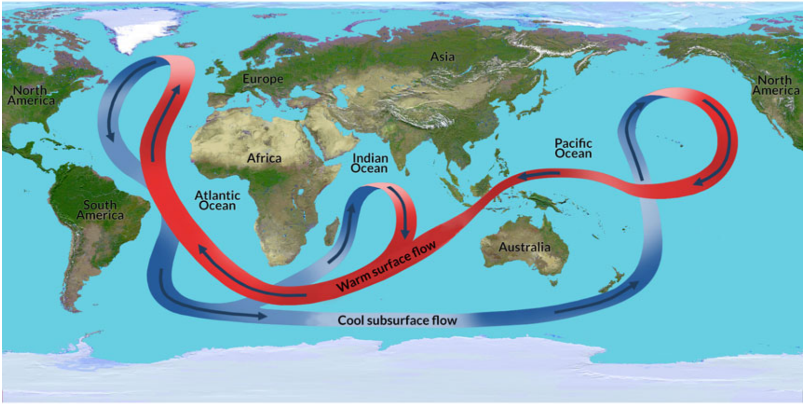

Figure: The Conveyor Belt (Broecker) indicating the global ocean circulation pattern. Source: JPL-CalTech/NASA

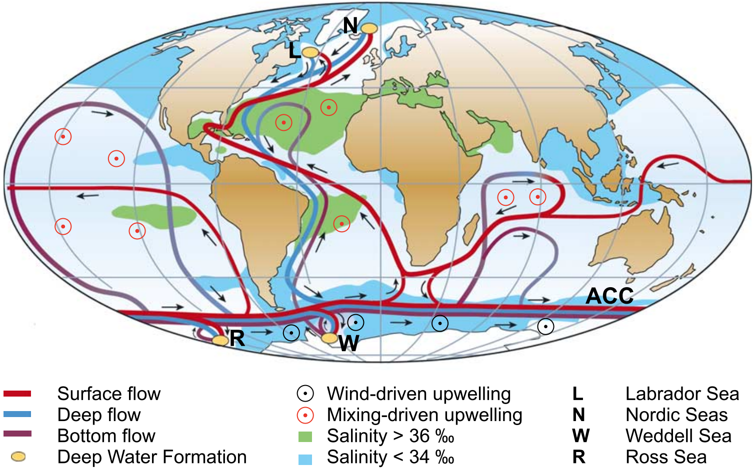

Figure: A more detailed sketch of the ocean circulation system including salinity concentrations (ACC = Antarctic Circumpolar Current). The entire loop takes many decades to complete. Source: “On the Driving Processes of the Atlantic Meridional Overturning Circulation,” Kuhlbrodt, et. al., *Reviews of Geophysics* **45** (2007) RG2001 (32 pp.).

Thermohaline Circulation (THC)

- Differences in density drive the flow in the oceans. Flow rate measured in sverdrups (Sv): 1 Sv = one million m^3/sec.
- The rate of ocean circulation is a function of temperature (thermo) and salinity (haline).
  1. The higher the salinity, the more dense the water.
  2. Cooler water is more dense than warmer water.

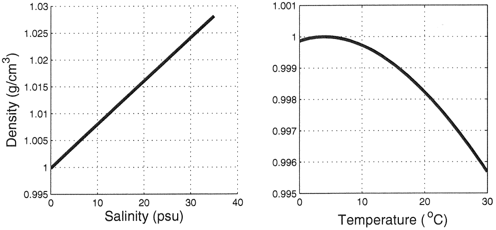

Figure 3.5. *Density as a function of salinity and temperature.*

Figure: *Mathematics and Climate*, Kaper and Engler, SIAM (2013), p. 33.

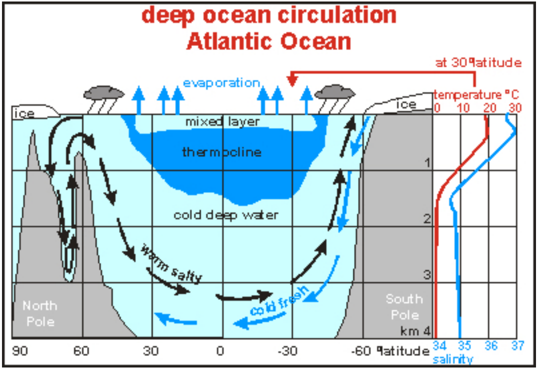

Figure: A sketch of a cross-section of the Atlantic Ocean as a function of latitude. The temperatures are essentially constant in the top mixed layer and the deeper abyssal zone (just above freezing). Note the absence of the mixed layer and thermocline near the poles, where nearly fresh ice is formed.
Source: “Oceanography: Currents and Circulation,” Anthoni, J. F., Seafriends (2000), http://www.seafriends.org.nz/oceano/current2.htm

An Advection-Diffusion Equation

The temperature in the thermocline region (between the top layer and the abyssal zone) is changed through advection (the transfer of heat from upwelling cold water) and diffusion from small-scale eddies.

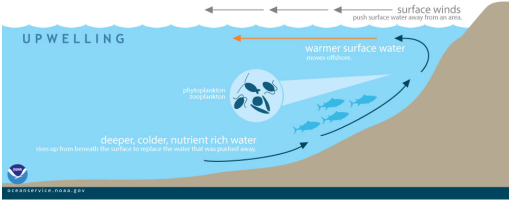

Figure: Upwelling: wind along the surface pushes water away allowing for colder water to rise up from below. Source: NOAA National Ocean Service

Model: Let $T = T(z, t)$ be the temperature at time t and depth z.

$$\frac{\partial T}{\partial t} = \omega \frac{\partial T}{\partial z} + c\frac{\partial^2 T}{\partial z^2} \quad \text{or} \quad T_t = \omega T_z + cT_{zz}$$

where ω = upwelling velocity and c = diffusion coefficient.

Stommel's Ocean Box Model

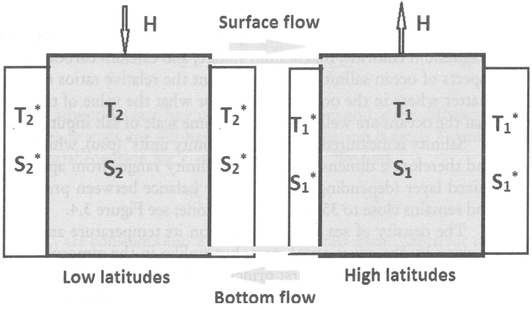

Figure: The two-box ocean model for temperature and salinity proposed by Henry Stommel in his paper "Thermohaline Convection with Two Stable Regimes," *Tellus* **XII** (1961), 224–230. Source: Kaper and Engler, p. 34.

$$\begin{aligned} T_i &= \text{temperature in box } i \\ T_i^* &= \text{surrounding temperature for box } i \\ S_i &= \text{salinity level in box } i \\ S_i^* &= \text{surrounding salinity level for box } i \end{aligned}$$

Stommel's Ocean Box Model

Model Assumptions:

- Density differences drive the flow between boxes: water in higher density box wants to flow toward lower density box. This flow happens through a pipe connecting boxes (bottom). The surface flow pipe at the top keeps the volume in each box constant.
- Boxes are assumed to be well-mixed so temperature and salinity are uniform throughout box (i.e., $T_i = T_i(t)$ and $S_i = S_i(t)$)
- The surrounding basins of each box (representing the atmosphere and neighboring oceans) are assumed to have constant temperatures T_i^* and salinity levels S_i^*.
- Heat and salinity are exchanged between each box and its surrounding basin.

One-Box Model

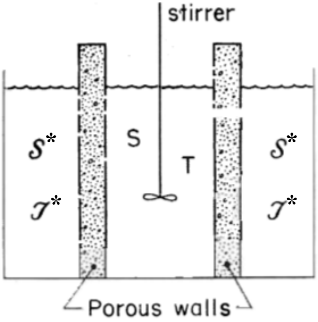

Figure from Stommel, *Tellus* **XII** (1961).

$$\frac{dT}{dt} = c(T^* - T)$$

$$\frac{dS}{dt} = d(S^* - S)$$

T^* and S^* are the constant temperature and salinity, respectively, of the surrounding fluid, while c and d are positive constants (rates).

Solution to One-Box Model

Solve each equation with separate and integrate technique:

$$S(t) = S^* + (S_0 - S^*)e^{-dt}$$
$$T(t) = T^* + (T_0 - T^*)e^{-ct}$$

For any initial condition (S_0, T_0), solution heads exponentially toward stable equilibrium (sink) at (S^*, T^*).

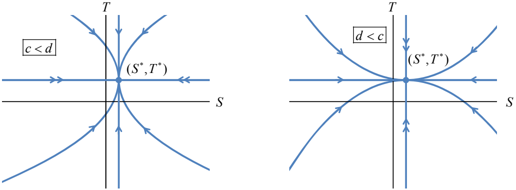

Figure: Phase portraits in the *ST*-plane. Solutions approach the sink tangent to the slower straight-line solution. Source: Dick McGehee, Univ. of Minnesota and MCRN, lecture slides.

Approximating Density

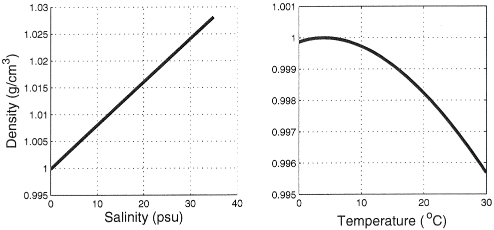

Figure 3.5. *Density as a function of salinity and temperature.*

Figure: Density (mass/volume) increases with salinity, but decreases with temperature. Source: *Mathematics and Climate*, Kaper and Engler, p. 33.

A linear approximation for density ρ:

$$\rho \;=\; \rho_0(1 - \alpha T + \beta S)$$

where ρ_0 is a reference density and α, β are positive constants.

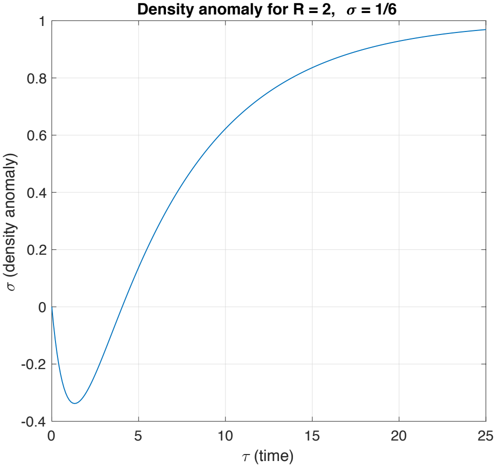

Figure: Plot of the density anomaly σ for the special solution with initial condition $x_0 = y_0 = 0$. At first the density decreases below the starting value ρ_0 (temperature more important, $\delta = 1/6$), but then density increases toward a value above ρ_0 as salinity effects take over ($R = 2$).

Stommel's Two-Box Model

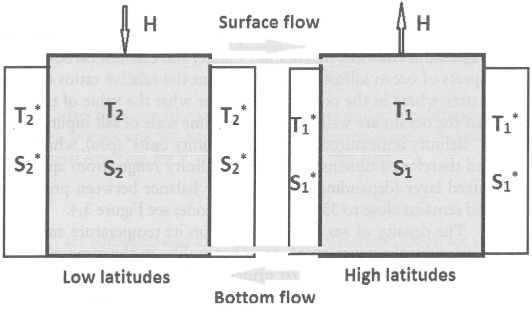

Figure: The two-box ocean model for temperature and salinity proposed by Henry Stommel in his paper "Thermohaline Convection with Two Stable Regimes," *Tellus* **XII** (1961), 224–230. Source: Kaper and Engler, p. 34.

$$
\begin{aligned}
T_i &= \text{temperature in box } i \\
T_i^* &= \text{surrounding temperature for box } i \\
S_i &= \text{salinity level in box } i \\
S_i^* &= \text{surrounding salinity level for box } i
\end{aligned}
$$

Four-Dimensional ODE Model

$$\begin{aligned} \dot{T}_1 &= c(T_1^* - T_1) + |q|(T_2 - T_1) \\ \dot{T}_2 &= c(T_2^* - T_2) + |q|(T_1 - T_2) \end{aligned} \qquad \begin{aligned} \dot{S}_1 &= d(S_1^* - S_1) + |q|(S_2 - S_1) \\ \dot{S}_2 &= d(S_2^* - S_2) + |q|(S_1 - S_2) \end{aligned}$$

q is the flow rate (signed) between the two tanks. Why $|q|$?

Answer: Flow is driven by differences in density $\rho_1 - \rho_2$, but which direction is defined as "positive" is irrelevant due to compensating surface flow.

- Suppose $S_1 > S_2$. Then water in tank 1 is more dense so flow moves from tank 1 toward tank 2 (q positive). Thus, the water in tank 2 becomes more salty (S_2 increases) while water in tank 1 is less salty (S_1 decreases). This agrees with model equations.
- Conversely, if $S_2 > S_1$, then water in tank 2 is more dense so flow moves in opposite direction (q negative). Now tank 2 becomes less salty (S_2 decreases) while tank 1 becomes more salty (S_1 increases). Need $|q|$ instead of q to insure this agrees with model.

Cutting the dimension in half

Define new variables u, v, T, S as follows:

$$u = \frac{1}{2}(T_1 + T_2), \qquad T = T_1 - T_2,$$
$$v = \frac{1}{2}(S_1 + S_2), \qquad S = S_1 - S_2.$$

In these variables, the system becomes

$$\dot{u} = c(u^* - u), \qquad \dot{T} = c(T^* - T) - 2|q|T,$$
$$\dot{v} = d(v^* - u), \qquad \dot{S} = d(S^* - S) - 2|q|S,$$

where $q = k\rho_0(-\alpha T + \beta S)$ and $T^* = T_1^* - T_2^*$, $S^* = S_1^* - S_2^*$.
The equations for u and v are easily solved, yielding

$$u(t) = u^* + (u_0 - u^*)e^{-ct}, \quad v(t) = v^* + (v_0 - v^*)e^{-dt},$$

thereby reducing the system from four dimensions to two.

Eliminating Parameters

As with the one-box model, define new variables and parameters

$$x = \frac{S}{S^*},\ y = \frac{T}{T^*},\ \delta = \frac{d}{c},\ \text{ and }\ \tau = c\,t.$$

New system becomes (HW)

$$\begin{aligned} x' &= \delta(1-x) - |f|x \\ y' &= 1 - y - |f|y \\ \lambda f &= -y + Rx, \end{aligned}$$

where

$$f = \frac{2q}{c},\ R = \frac{\beta S^*}{\alpha T^*},\ \lambda = \frac{c}{2k\rho_0\alpha T^*},\ \text{ and }\ ' = \frac{d}{d\tau}.$$

f is the new flow rate and λ is a measure of the strength of the flow.

Two-dimensional ODE (coupled) with three parameters (λ, R, δ).

Equilibrium Points

Define the function $G(f; R, \delta) = \dfrac{R\delta}{\delta + |f|} - \dfrac{1}{1 + |f|}$.

For a fixed value of λ, suppose that f satisfies $\lambda f = G(f)$. Then

$$(x, y) \ = \ \left(\frac{\delta}{\delta + |f|}, \ \frac{1}{1 + |f|}\right)$$

is an equilibrium point.

Solutions to $\lambda f = G(f)$ can be located graphically.

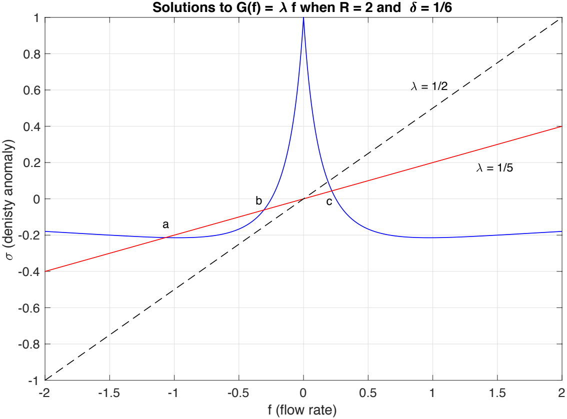

Figure: Solutions to the equation $G(f) = \lambda f$ when $R = 2$ and $\delta = 1/6$. If $\lambda = 1/2$ (dashed black), there is only one solution (and thus only one equilibrium point). But if $\lambda = 1/5$ (red), there are three solutions $f_1 \approx -1.0679$, $f_2 \approx -0.30703$, and $f_3 \approx 0.21909$ corresponding to three equilibria.

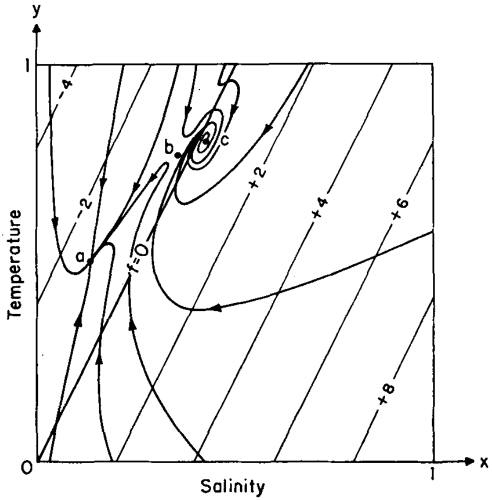

Figure: The phase plane for Stommel's reduced two-box model for parameter values $R = 2, \delta = 1/6$, and $\lambda = 1/5$. There are three equilibria: a is a sink, b is a saddle, and c is a spiral sink. Source: "Thermohaline Convection with Two Stable Regimes," H. Stommel, *Tellus* **XII** (1961), 224–230.

Interpretation of Stable Equilibria

Recall: $f = \frac{2q}{c}$ and $q = k(\rho_1 - \rho_2) = \rho_0(-\alpha T + \beta S)$. $T = T_1 - T_2$ and $S = S_1 - S_2$ are temperature and salinity differences, respectively, between the two tanks.

- At equilibrium point a, $f < 0$ so $q < 0$. This implies $\rho_2 > \rho_1$ so flow is going from tank 2 to tank 1. Since $q < 0$, temperature differences are more important than salinity differences. Flow moves from colder to warmer tank, even though tank 1 has higher salinity levels ($S_1 > S_2$).

- At equilibrium point c, $f > 0$ so $q > 0$. This implies $\rho_1 > \rho_2$ so flow is going from tank 1 to tank 2. Since $q > 0$, salinity differences are more important than temperature differences. Flow moves from warmer to colder tank ($T_1 > T_2$).

The two equilibria have opposite flow directions.

Implications for Climate System

> *The fact that even in a very simple convective system, such as here described, two distinct stable regimes can occur ... suggests that a similar situation may exist somewhere in nature. One wonders whether other quite different states of flow are permissible in the ocean or some estuaries and if such a system might jump into one of these with a sufficient perturbation. If so, the system is inherently frought with possibilities for speculation about climatic change.*

Stommel, "Thermohaline Convection with Two Stable Regimes," p. 228.

Bifurcation: If λ becomes large enough, system loses two equilibria and a solution could jump from equilibrium point a to c, flipping its flow direction and changing the primary mechanism driving the flow from temperature to salinity.