MATH 392-02: Seminar in Complex Analytic Dynamics
Homework Assignment #5

DUE DATE: Mon., April 2, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. You are encouraged to work on these problems with other classmates, although the solutions you turn in should be your **own** work. Please cite any references (websites, texts, etc.) that you may have used for assistance with the assignment.

Note: Please list the names of any students or faculty who you worked with on the top of the assignment.

1. In this problem you will give two different proofs of an important fact needed to finish the proof of the Lemma that, in turn, is used to prove that the Julia set of a rational map of degree $d \geq 2$ is a perfect set. Suppose that z_0 is a period n point for R and let $S = R^n$. Furthermore, suppose that the equation

$$S(z) - z_0 \; = \; 0 \tag{1}$$

has **exactly** one solution, that being $z = z_0$. Show that z_0 is a super-attracting fixed point for S (and thus a super-attracting period n point for R) in each of the following two ways:

a) Write $S(z) = p(z)/q(z)$ and show directly that $S'(z_0) = 0$.

b) Define the Möbius transformation $M(z) = 1/(z - z_0)$ and consider the conjugate map $g = M \circ S \circ M^{-1}$ on $\overline{\mathbb{C}}$. Show that $g(z)$ has no poles in $\mathbb{C}$ and conclude that g is a polynomial. Now use the conjugate fixed point theorem.

2. Recall that the set of *exceptional points* $E(R)$ for a rational map R is the set $\overline{\mathbb{C}} - \cup_{n=0}^{\infty} R^n(U)$ where U is any neighborhood of a point $z \in J(R)$. The definition of $E(R)$ is independent of which point z is chosen from the Julia set. $E(R)$ contains at most two points and these points are necessarily in the Fatou set.

a) Show that ∞ is an exceptional point for any polynomial $p(z)$.

b) Suppose that z_0 is a fixed point and that $R^{-1}(z_0) = \{z_0\}$, that is, the only pre-image of z_0 is itself. Show that $z_0 \in E(R)$.

3. Let z be an arbitrary point in the Julia set of $R(z)$ and let

$$B = \bigcup_{n=0}^{\infty} R^{-n}(z).$$

B is the set of all pre-images of z under the map R and its iterates. Show that $J(R) = \overline{B}$, where $\overline{B}$ denotes the closure of the set B. This gives a simple algorithm for using a computer to sketch the Julia set.

4. Recall that the *Cantor Middle-Thirds Set* Λ is obtained by starting with the closed unit interval $[0, 1]$ and deleting the open middle third $(1/3, 2/3)$. From the remaining two intervals, delete the next two open middle-thirds, $(1/9, 2/9) \cup (7/9, 8/9)$. The process continues again, deleting the next four open middle-thirds from the remaining four intervals. The set that remains after repeating this process infinitely often is the Cantor Middle-Thirds Set Λ. Show that Λ is a perfect set.

5. Suppose that $R(z) = p(z)/q(z)$ is a rational map of degree d. Show that the number of critical points of $R(z)$ (including the possibility that ∞ is a critical point) is at most $2d - 2$. Note that ∞ is a critical point if and only if the conjugate map $g(z) = 1/R(1/z)$ satisfies $g'(0) = 0$. *Suggestion:* Break the problem up into different cases.

6. Suppose that z_0 is an attracting fixed point for $R(z)$. Moreover, suppose that $S(z)$ is an analytic inverse of $R(z)$ defined on a neighborhood U of z_0 with $S(z_0) = z_0$. What type of fixed point is z_0 for the function S? Prove your assertion.