Mathematical Models
MATH 303 Fall 2018

Midterm Exam

Friday, Oct. 5, in class

The midterm exam covers Chapters 1–3 of the text by K. K. Tung, the material on the first three homework assignments, and the first computer project. It is highly recommended that you go over homework problems and your class notes.

Exam Review: We will review for the exam during class on Wednesday, Oct. 3. Please come prepared with specific questions.

Note: You will be need a scientific calculator for the exam which does NOT have graphing or symbolic capabilities. Please bring your own calculator with you to the exam.

The following topics, definitions and theorems are important material for the exam.

- The Fibonacci Numbers: appearance in nature, use in art and music, recursive definition, identities, connection to the Golden Ratio
- The Golden Ratio: definition of, derivation of, connection to Fibonacci numbers, use in art, architecture, and music, continued fraction expansion of
- Working with Data: fitting data to a curve, linear regression, the method of least squares, R^2 value (meaning of), log-log plots
- Scaling Laws: Kleiber's 3/4 power law, derivation of (know the basic outline and key assumptions), Zipf's law (examples of), scaling laws in networks (e.g., world wide web)
- Compound Interest and Mortgage Payments: compounding interest over different time periods (e.g., annually, monthly, daily), compounding continuously, modeling with linear difference equations, computing mortgage payments
- Mathematical Techniques: proof by induction, limits, sum of a finite and infinite geometric series, solving first- and second-order linear difference equations

Some Sample Exam Questions

1. Recall that the Lucas numbers $\{L_n\}_{n=0}^{\infty}$ satisfy the same recursive relation as the Fibonacci numbers, but begin with $L_0 = 1$ and $L_1 = 3$.

   (a) Write out the first 10 Lucas numbers.

   (b) What is $\lim_{n\to\infty} L_{n+1}/L_n$? What is $\lim_{n\to\infty} L_{n+3}/L_n$?

   (c) Find an explicit formula for the nth Lucas number.

   (d) Find a formula for $L_n L_{n+2} - L_{n+1}^2$ and prove it by induction.

2. Describe three key assumptions in the derivation of Kleiber's 3/4 power law by West, Brown, and Enquist. Which assumption do you find to be the most controversial?

3. Find the value of the continued fractions α and β:

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} \qquad\qquad \beta = 1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{\ddots}}}}$$

4. Consider the first-order linear difference equation

$$y_{k+1} = -\frac{1}{4}y_k + \frac{3}{4}, \qquad y_0 = 7\,.$$

   (a) Write out the first 6 terms of the sequence generated by the recursion.

   (b) Find and simplify the solution y_k. Check your formula works for the first few values of the sequence.

   (c) Compute $\lim_{k\to\infty} y_k$.

   (d) Find the value of y_0 that is fixed under the recursive relation, that is, the resulting sequence is simply $y_0, y_0, y_0, \ldots$.

5. Find the solution to the second-order difference equation

$$y_{k+2} = 2y_{k+1} + 15y_k$$

   satisfying the initial conditions $y_0 = 0$ and $y_1 = 1$. Write out the first 6 terms of the sequence and check a few against your formula.

6. Consider the second-order difference equation

$$y_{k+2} = 2y_{k+1} - y_k$$

   satisfying the initial conditions $y_0 = 1$ and $y_1 = 2$.

   (a) Write out the first 6 terms of the sequence generated by the recursion.

   (b) Find the solution y_k and verify that it satisfies the difference equation.

   (c) Find the general solution to the difference equation in terms of arbitrary initial conditions y_0 and y_1, that is, find a formula for y_k that depends only on k, y_0, and y_1.

7. Working with mortgages and compound interest.

   (a) Suppose you borrow \$350,000 on a 15-year mortgage at an annual interest rate of 3% compounded monthly. What is your monthly mortgage payment?

   (b) Under the same conditions as **(a)**, how much does your monthly mortgage payment increase if the compounding is done daily rather than monthly?

   (c) You take out a student loan for \$120,000 to pay for medical school. You are charged 5% annual interest, compounded continuously and decide to pay \$1,000 every month toward the loan. How long will it take you to pay the entire loan off, including interest? How much total interest will you pay?

(d) Is it better to receive \$500 today or \$600 in 5 years if the interest rate is 3%? What if the rate increases to 4%? Assume that interest is compounded continuously.

(e) Congratulations, you just won \$2 million dollars in the lottery! However, you do not get all of your money now; you will receive four yearly payments of \$500,000 beginning immediately. Assuming an interest rate of 4% compounded continuously, what is the present value of your prize? How much do you "lose" by not receiving the full prize today?

Note: The present value of P is the amount you would have to invest *now* in order to have P dollars t years in the future.