MATH 242: Principles of Analysis (Fall 2009)
Practice Problems for Exam 2

1. Construct a divergent sequence that has three convergent subsequences with distinct limits.

2. Construct a bounded sequence that is not Cauchy.

3. Determine whether the given infinite series converges or diverges using any of the tests from class or the text. Be sure to verify the hypotheses of the test you are applying.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} e^{1/n^2}$

4. Show that the sum of the lengths of the deleted intervals in the Cantor set is one.

5. Find $\bar{A}$ for each of the following sets:

   (a) $A = (0, \infty)$.

   (b) $A = \{\sqrt{2} + \frac{1}{n} : n \in \mathbb{N}\}$

   (c) $A = \mathbb{Q}$

6. For each of the sets below, decide which of the following attributes apply to the set: open, closed, bounded, compact, or connected

   (a) $A = (0, \infty)$.

   (b) $A = \mathbb{Z}$

   (c) $A = \{n \in \mathbb{N} : n^2 < 101\}$

   (d) Cantor set

7. Prove that $\displaystyle\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist.

8. Find the following limits and then use the ϵ-δ definition to prove your claim.

   (a) $\lim_{x\to 2} (3x^2 + 8x)$

   (b) $\lim_{x\to 1} \dfrac{1}{5x-3}$

   (c) $\lim_{x\to 0} x^2 \sin\left(\dfrac{1}{x}\right)$

9. Construct a function that is not continuous at the points $x = 0$ and $x = 1$, but is continuous everywhere else.

10. Construct a function that is continuous at the points $x = 0$ and $x = 1$, but is not continuous everywhere else.

11. Suppose that f is a continuous function that maps $[0, 1]$ into $[0, 1]$. Take any point $c \in [0, 1]$ and define the recursive sequence $x_1 = c$ and $x_{n+1} = f(x_n)$ for any $n \in \mathbb{N}$. Suppose that $\{x_n\}$ converges to the point L. Prove that L is a fixed point of f, that is, show that $f(L) = L$.

12. Show that there are at least three distinct solutions to the equation $2\sin x + \cos x = x$.

13. True or False: If true give a proof (or reference the appropriate theorem), if false give a counterexample or correct the statement.

   a) If $\lim_{n\to\infty} a_n = 0$, then the infinite series $\sum_{n=1}^{\infty} a_n$ converges.

   b) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ also converges.

   c) All monotone sequences are Cauchy.

   d) Every Cauchy sequence converges.

   e) The irrationals are a disconnected set.

   f) If f is continuous on $\mathbb{R}$, then its range must be a connected set.

   g) Any polynomial of even degree with a real root has at least two distinct real roots.

   h) If $f : [0, 1] \to [0, 1]$, then f attains a maximum on $[0, 1]$.