MATH 242 Principles of Analysis

Exam #2 Solutions **November 19, 2009** **Prof. G. Roberts**

1. Quickies. No work or explanation required. (18 pts.)

   (a) Give an example of an unbounded sequence that contains a subsequence that is Cauchy.
   Solution: $\{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \ldots\}$

   (b) Give an example of a set K that is compact but is not connected.
   Solution: $K = [0, 1] \cup [2, 3]$ works as does the Cantor set or any finite set.

   (c) Give an example of an infinite series $\sum_{n=1}^{\infty} a_n$ that diverges yet has $\lim_{n\to\infty} a_n = 0$.

   Solution: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

2. Let $A = \mathbb{Q} \cap (0, 1)$

   (a) Which of the following attributes apply to A? (circle all that apply) (8 pts.)

   $\boxed{\textbf{i)}}$ bounded **ii)** open **iii)** compact **iv)** connected

   Solution: The set A is bounded because it is a subset of $(0, 1)$. It is not open because any ϵ-neighborhood of a rational contains an irrational number (since the irrationals are dense in $\mathbb{R}$). It is not closed because it does not contain all its limit points. Any irrational number q in $(0, 1)$ is a limit point of A because a sequence of rationals can be constructed that converges to q. Therefore, A is not compact (by the Heine-Borel Theorem). Finally, A is not connected because it is not an interval.

   (b) Find $\bar{A}$. (5 pts.)
   Solution: As argued above, any irrational in $(0, 1)$ is a limit point of A. The sequences $x_n = 1/(2n)$ and $y_n = 1 - 1/(2n)$ are each sequences in A converging to 0 and 1, respectively. Thus, 0 and 1 are also limit points of A. Therefore,

   $$\bar{A} \;=\; A \cup (\mathbb{I} \cap (0, 1)) \cup \{0, 1\} \;=\; [0, 1].$$

3. Determine whether the given infinite series converges or diverges using any of the tests from class or the text. In each case, be sure to verify the hypotheses of the test you are applying. (18 pts.)

   (a) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5 + 2}}$
   Solution: This series converges by the comparison test. We have, for any $n \in \mathbb{N}$,

   $$0 \;\leq\; \frac{n}{\sqrt{n^5 + 2}} \;\leq\; \frac{n}{n^{5/2}} \;=\; \frac{1}{n^{3/2}}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-series test ($p = 3/2$), the original, smaller series, converges by the comparison test.

(b) $\sum_{n=1}^{\infty} \frac{n-1}{n+1}$

Solution: This series diverges by the nth term test since $\lim_{n\to\infty} a_n = 1$.

(c) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

Solution: This series converges by the alternating series test. We have that $\frac{1}{\ln n} > 0$ since $n \geq 2$. Since $\ln x$ is an increasing function, the sequence $\{\ln n\}$ is an increasing, positive sequence and thus, $\{\frac{1}{\ln n}\}$ is a decreasing sequence. Finally, since $\lim_{n\to\infty} \ln n = \infty$ we have $\lim_{n\to\infty} \frac{1}{\ln n} = 0$. This verifies all the hypotheses of the alternating series test, showing that the series converges.

4. Use the ϵ-δ definition of the limit of a function to prove the following limit: (15 pts.)

$$\lim_{x\to -1} (2x^2 - x + 5) \ = \ 8$$

Solution: Preamble: We compute that

$$|f(x) - L| \ = \ \left|2x^2 - x + 5 - 8\right| \ = \ \left|2x^2 - x - 3\right| \ = \ |2x-3| \cdot |x+1| \ = \ |2x-3| \cdot |x - (-1)|.$$

We can make the $|x+1|$ term as small as we like by choosing δ appropriately. We need to bound the $|2x-3|$ term. To do this, set $\delta = 1$. Then $0 < |x+1| < 1$ implies $-2 < x < 0$. On this interval, the function $|2x-3|$ is decreasing and has a supremum of 7. Thus we can bound $|f(x) - L|$ by $7|x+1|$. Choosing $\delta \leq \epsilon/7$ will then suffice.

Proof: Let $\epsilon > 0$ be given. Choose $\delta = \min\{1, \epsilon/7\}$. Then,

$$\begin{aligned}
\left|2x^2 - x + 5 - 8\right| &= \left|2x^2 - x - 3\right| \\
&= |2x-3| \cdot |x+1| \\
&< 7 \cdot |x+1| \quad \text{since } \delta \leq 1 \\
&< 7 \cdot \epsilon/7 \quad \text{since } \delta \leq \epsilon/7 \\
&= \epsilon \quad \text{whenever } 0 < |x+1| < \delta . \qquad \square
\end{aligned}$$

5. Consider the function $f(x)$ defined as $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$

(a) Is f continuous at $x = 0$? Justify your claim. (8 pts.)

Solution: No. Let $x_n = \frac{\sqrt{2}}{n}$. Since x_n is the product of a rational and an irrational number, it is irrational for each $n \in \mathbb{N}$ (proved on HW). Therefore, we have $x_n \to 0$ and $f(x_n) = x_n \to 0$ but $f(0) = 1$. Since $f(x_n) \not\to f(0)$, f is not continuous at 0.

(b) Is f continuous at $x = 1$? Justify your claim. (8 pts.)

Solution: Yes, because $\lim_{x\to 1} f(x) = f(1)$.

Let $\epsilon > 0$ be given. Set $\delta = \epsilon$. Then

$$|f(x) - f(1)| \;=\; |f(x) - 1| \;=\; \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ |x-1| & \text{if } x \notin \mathbb{Q} \end{cases}$$

In either case, we find that $|f(x)-1| < \epsilon$ whenever $0 < |x-1| < \delta$. This proves rigorously that $\lim_{x\to 1} f(x) = f(1)$, as desired.

6. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, provide a **counterexample** or justification for your choice. (10 pts. each)

(a) The sequence $\{\sin n\}$ is divergent, but contains a convergent subsequence.

Solution: TRUE. First, the sequence $\{\sin n\} = \{\sin 1, \sin 2, \sin 3, \ldots\}$ bounces around the interval $(-1, 1)$, never approaching a unique value. It is not a Cauchy sequence since $|\sin(n+1) - \sin(n)|$ never becomes arbitrarily small. Therefore, it does not converge. Second, $|\sin n| \le 1 \; \forall n \in \mathbb{N}$, so the sequence is bounded. By the Bolzano-Weierstrauss Theorem, it must contain a convergent subsequence. $\square$

(b) If f is a continuous function on $[a, b]$, then $f([a, b])$ is a bounded and closed interval.

Solution: TRUE. By the Heine-Borel Theorem, $[a, b]$ is a compact set. Since f is continuous, it takes compact sets to compact sets. Therefore, $f([a, b])$ is a compact set and is closed and bounded. Moreover, since f is continuous, it takes connected sets to connected sets. Since connected sets in $\mathbb{R}$ are intervals, it follows that $f([a, b])$ is a bounded and closed interval. $\square$