MATH 241-02, Multivariable Calculus, Spring 2019

Section 9.1: Three-Dimensional Coordinate Systems

Before we can study functions of more than one variable, we need to describe a space large enough to contain both the domain and the range of a function. For example, to visualize a function of two variables $z = f(x, y)$, we need a three-dimensional coordinate system: two dimensions for the input variables x and y, and a third dimension for the output variable z. In this section we learn about the standard xyz coordinate system and how to describe some basic geometric shapes (e.g., plane, sphere) in these coordinates.

The xyz Coordinate System

The standard way to draw a three-dimensional coordinate system is shown in Figure 1. On the left are the three **coordinate axes**, the x-axis, y-axis, and z-axis, each perpendicular to the other. They intersect at the **origin** (denoted by O), which has coordinates $(0, 0, 0)$. The arrows indicate the positive direction on each axis. It is very important to label the axes in the way shown in the figure. The axes determine three **coordinate planes**, as shown on the right in Figure 1 (e.g., the xz-plane contains the x- and z-axes). Two coordinate planes intersect in one of the coordinate axes, and all three planes meet at the origin. One way to visualize all of this is to regard the corner of a room as the origin and then each wall is a different coordinate plane.

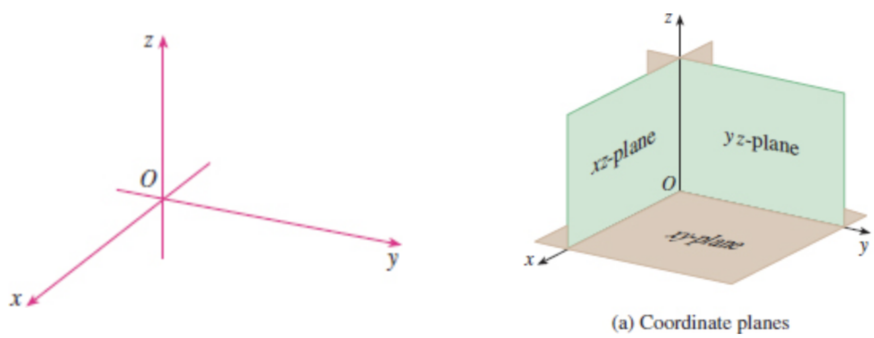

Figure 1: The coordinate axes (left) and coordinate planes (right).

The mathematical notation for three-space is $\mathbb{R}^3$ which represents three copies of the real line, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. This is called the **Cartesian product**. The real line is just $\mathbb{R}$ while the xy-plane is $\mathbb{R}^2$. To describe a point in three dimensional space, we need three numbers or an **ordered triple** (a, b, c). The point (a, b, c) means a point that is a units in the x direction, b units in the y direction, and c units in the z-direction (see Figure 2).

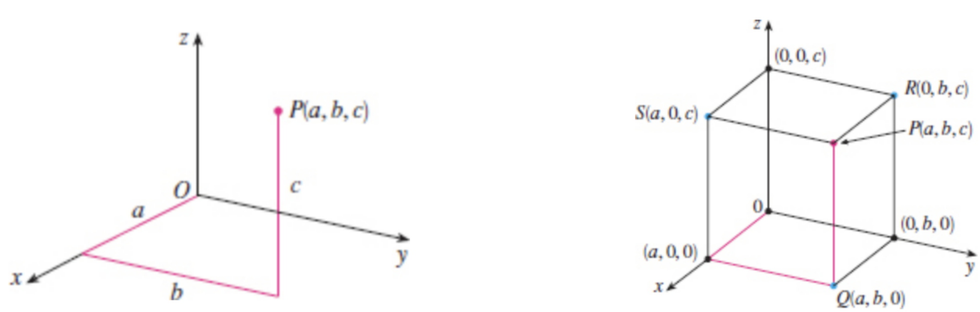

Figure 2: Plotting points in $\mathbb{R}^3$.

Exercise 1: Use a rectangular box, as shown to the right in Figure 2, to plot the point $(-2, 4, 1)$.

Note that a point on the x-axis has coordinates of the form $(a, 0, 0)$ for some $a \in \mathbb{R}$. Similarly, any point on the y-axis is given by $(0, b, 0)$, and any point on the z-axis is represented by $(0, 0, c)$. Any point in the xy-plane will have a z-coordinate equal to zero (the point can have no "height"). Hence, the equation of the xy-plane is simply $z = 0$ (the values of x and y are arbitrary). Likewise, the xz-plane has equation $y = 0$ and the yz-plane has equation $x = 0$.

Important Note: When working in three-dimensional space with an equation that is missing one or more variables, always assume that the values of those variables are *arbitrary*. Just because the variables are missing does not mean you should ignore them!

Exercise 2: Sketch and describe the set of all points satisfying the given equation or inequality:

(a) $y = 3$ **(b)** $x^2 + y^2 = 1$ **(c)** $x^2 + y^2 = 1$, $-1 \leq z \leq 1$

Distance Between Two Points

Recall that in two dimensions, the distance between two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This formula comes from drawing a right triangle between the two points and applying the Pythagorean Theorem. The same approach works in three dimensions except the picture is a little more complicated (see Figure 3).

Distance Formula: The distance between the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \, .$$

Exercise 3: Find the distance between the points $(2, -5, 4)$ and $(4, 0, -2)$.

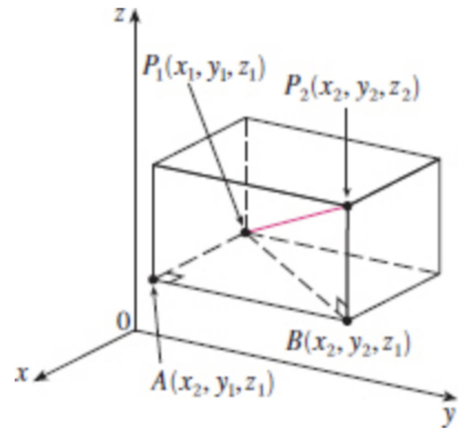

Figure 3: The distance between two points in xyz-space can be obtained with two applications of the Pythagorean Theorem. First we have $|P_1B|^2 = |P_1A|^2 + |AB|^2$, then we have $|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$.

Recall that a sphere is the set of all points a fixed distance r (the radius) from a given point (the center).

Exercise 4: Find the equation of a sphere centered at $(2, -1, 0)$ with radius 3.

Hint: Let (x, y, z) be a random point on the sphere and use the distance formula to find an equation that x, y, and z must satisfy.

Based on the previous problem, the general formula for a sphere of radius r centered at (a, b, c) is

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 \ = \ r^2 \ .}$$

The **unit sphere** is the special sphere centered at the origin with radius 1. Letting $a = b = c = 0$ and $r = 1$ in the previous formula, we find that

$$\boxed{x^2 + y^2 + z^2 \ = \ 1}$$

is the formula for the unit sphere. Notice how this generalizes the formula for the unit circle $x^2 + y^2 = 1$ in the xy-plane.

Exercise 5: Show that the equation $2x^2 + 2y^2 + 2z^2 + 4x - 6y + 5z = 0$ describes a sphere in xyz-space, and find the center and radius of the sphere.

Exercise 6: Describe the intersection of the sphere $x^2 + y^2 + z^2 = 8$ and the plane $y = 2$. Give the equation(s) of the intersection.

Exercise 7: Suppose that the Earth was given a coordinate system that matched the unit sphere (i.e., the center of the Earth is taken to be the origin and the radius is assumed to be 1.)

(a) What are the coordinates of the North pole?

(b) What are the coordinates of the South pole?

(c) What are the coordinates of any point on the equator?

(d) What are the coordinates of any point on the Arctic circle?

(e) Find a point on the unit sphere where each coordinate is a nonzero rational number. A rational number is one that can be written as the ratio of two integers.