MATH 241-02, Multivariable Calculus, Spring 2019

Section 10.4: Motion in Space: Velocity and Acceleration

This section focuses on the velocity and acceleration of a moving object in 3D space. The ideas build on the material from previous sections. We will learn how to decompose the acceleration vector into its tangential and normal components.

Velocity, Speed, and Acceleration

Let $\mathbf{r}(t)$ represent the position of a particle (or object) traveling through space at time t. The **velocity** of the particle is given by the tangent vector, $\mathbf{v}(t) = \mathbf{r}'(t)$. This vector points in the direction of motion of the particle. The length of this vector, $|\mathbf{r}'(t)|$, is the **speed** of the particle. Finally, the **acceleration** of the particle is the derivative of velocity, or the second derivative of position, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example 1: Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = < e^{3t},\, 4t^2 + 1,\, \sin(2t) >$.

Answer: The velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t) = < 3e^{3t},\, 8t,\, 2\cos(2t) >$ and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = < 9e^{3t},\, 8,\, -4\sin(2t) >$. The speed is found by taking the length of the velocity vector:

$$|\mathbf{v}(t)| \;=\; \sqrt{(3e^{3t})^2 + (8t)^2 + (2\cos(2t))^2} \;=\; \sqrt{9e^{6t} + 64t^2 + 4\cos^2(2t)}\,.$$

Exercise 1: Suppose that a particle has an initial position of $\mathbf{r}(0) = 3\,\mathbf{i} - \mathbf{k}$ and an initial velocity of $\mathbf{v}(0) = 2\,\mathbf{i} + \mathbf{j} + 4\,\mathbf{k}$. If its acceleration is $\mathbf{a}(t) = 5t\,\mathbf{i} + e^{-2t}\,\mathbf{j} + 4\,\mathbf{k}$, find its velocity $\mathbf{v}(t)$ and position $\mathbf{r}(t)$.

Decomposing Acceleration into its Tangential and Normal Components

In order to better understand the meaning of the acceleration vector, it is useful to decompose it in terms of the tangential vector $\mathbf{T}$ and normal vector $\mathbf{N}$. It turns out that there is no component in the direction of the binormal vector $\mathbf{B}$.

Recall that

$$\mathbf{T}(t) \;=\; \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \;=\; \frac{\mathbf{v}(t)}{|\mathbf{r}'(t)|} \quad\Longrightarrow\quad |\mathbf{r}'(t)|\,\mathbf{T}(t) \;=\; \mathbf{v}(t)\,.$$

If we differentiate both sides of the last equation with respect to t using the product rule (which holds for a scalar function times a vector function), we find

$$\frac{d}{dt}\left(|\mathbf{r}'(t)|\right)\mathbf{T}(t) + |\mathbf{r}'(t)|\,\mathbf{T}'(t) \;=\; \mathbf{a}(t)\,. \tag{1}$$

Next, using the formulas

$$\mathbf{N}(t) \;=\; \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \quad \text{and} \quad \kappa(t) \;=\; \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}\,,$$

we have

$$\mathbf{T}'(t) \;=\; |\mathbf{T}'(t)|\,\mathbf{N}(t) \;=\; \kappa(t)|\mathbf{r}'(t)|\,\mathbf{N}(t)\,.$$

Substituting this expression into equation (1), we find that

$$\mathbf{a}(t) \;=\; \frac{d}{dt}\left(|\mathbf{r}'(t)|\right)\mathbf{T}(t) + \kappa(t)|\mathbf{r}'(t)|^2\,\mathbf{N}(t)\,.$$

We have proven the following fact.

Let $v = |\mathbf{r}'(t)|$ be the speed of a particle. The acceleration vector can be written as

$$\mathbf{a} \;=\; a_T\mathbf{T} + a_N\mathbf{N}\,,\ \text{where } a_T = v',\ a_N = \kappa v^2\,.$$

The scalars a_T and a_N are the tangential and normal components of acceleration, respectively.

Figure 1 demonstrates the decomposition of $\mathbf{a}(t)$ into its tangential and normal components. Our argument above shows that there is no component of the acceleration vector in the binormal direction ($\mathbf{a}(t)\cdot\mathbf{B}(t) = 0\ \forall t$). In other words, $\mathbf{a}$ is always in the plane determined by $\mathbf{T}$ and $\mathbf{N}$. The tangential component of the acceleration is the derivative of the speed while the normal component is the product of the curvature and the speed squared. This is why you feel such a strong force when taking a sharp turn in a car. A large curvature implies a higher value of a_N which means a stronger force in the direction perpendicular to the motion (recall $\mathbf{F} = m\mathbf{a}$). At high speeds this effect is even more pronounced because of the v^2 term.

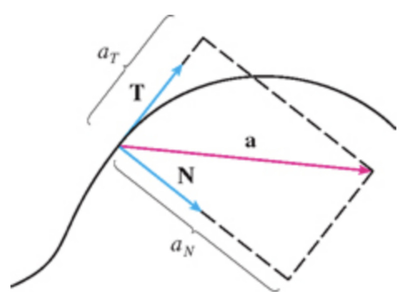

Figure 1: The acceleration vector always lies in the same plane as $\mathbf{T}$ and $\mathbf{N}$, and can thus be written as a linear combination of those vectors.

Recall that

$$\kappa(t) \;=\; \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

was one of our formulas for curvature. Since $a_N(t) = \kappa(t)v(t)^2 = \kappa(t)|\mathbf{r}'(t)|^2$, we see that

$$\boxed{a_N(t) \;=\; \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}\,.}$$

This formula is easier to use in most applications.

Exercise 2: Explain why any particle traveling at constant speed will always have an acceleration vector that is orthogonal to the direction of motion. Check that this is the case for the standard helix (see Exercise 5 from the worksheet for Section 10.3).

Exercise 3: Find the tangential a_T and normal a_N components of the acceleration vector for a particle with position function $\mathbf{r}(t) = < t,\ t^2,\ t^3 >$. Do this for any time t and for time $t = 0$. The curve traced out by $\mathbf{r}(t)$ is called the **twisted cubic.**

Exercise 4: The position function of the *USS Starship Enterprise* is given by

$$\mathbf{r}(t) \; = \; (3+t)\,\mathbf{i} + (2+\ln t)\,\mathbf{j} + \left(7 - \frac{4}{t^2+1}\right)\mathbf{k}\,,$$

where t is measured in minutes. A space station is located at the point $(6, 4, 9)$ and Captain Picard would like the *Enterprise* to coast into the station. At what time should the engines be turned off? Assuming the speed of the ship is constant once the engines are off, how long will it take the ship to reach the station?