MATH 136-03 Calculus 2, Spring 2019

Section 7.2: Trigonometric Integrals

This section focuses on integrals involving powers of $\cos x$ and $\sin x$. For these types of integrals, the key is to use the correct trig identity to simplify the integrand into a form where a u-sub or formula may be applied to compute the integral.

Odd Powers of $\cos x$ or $\sin x$

Example 1: Compute $\displaystyle\int \cos^5 x\, dx$ using the identity $\cos^2 x = 1 - \sin^2 x$.

Answer: When either $\cos x$ or $\sin x$ is raised to an **odd** power, break off one of the $\cos x$ or $\sin x$ terms and then use the identity above to replace the even power. For this example, we write

$$\cos^5 x \;=\; \cos x \cdot \cos^4 x \;=\; \cos x \cdot (\cos^2 x)^2 \;=\; \cos x (1 - \sin^2 x)^2 \;=\; (1 - 2\sin^2 x + \sin^4 x)\cos x.$$

Now we can use a u-substitution with $u = \sin x$ and $du = \cos x\, dx$. We obtain

$$\int \cos^5 x\, dx \;=\; \int (1 - 2\sin^2 x + \sin^4 x)\cos x\, dx \;=\; \int 1 - 2u^2 + u^4\, du \;=\; u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c.$$

Returning to the original variable, the solution is $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$.

A similar strategy will work when $\sin x$ is raised to an odd power. In this case, use the identity $\sin^2 x = 1 - \cos^2 x$ and let $u = \cos x$.

Even Powers of $\cos x$ and $\sin x$

When **both** $\cos x$ and $\sin x$ are raised to an even power, we can use one of the double angle formulas

$$\boxed{\cos^2 x \;=\; \frac{1}{2}\left(1 + \cos(2x)\right) \quad \text{or} \quad \sin^2 x \;=\; \frac{1}{2}\left(1 - \cos(2x)\right).} \tag{1}$$

These identities can be derived from the double angle formulas for cosine:

$$\cos(2x) \;=\; \cos^2 x - \sin^2 x \;=\; 2\cos^2 x - 1 \;=\; 1 - 2\sin^2 x.$$

Note that the last two expressions can be obtained from the first by using $\cos^2 x + \sin^2 x = 1$.

Example 2: Evaluate $\displaystyle\int_0^{\pi/4} \cos^2\theta\, d\theta$.

Answer: We use the first equation in (1).

$$\int_0^{\pi/4} \cos^2\theta\, d\theta \;=\; \int_0^{\pi/4} \frac{1}{2}(1+\cos(2\theta))\, d\theta \;=\; \frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta))\Big|_0^{\pi/4} \;=\; \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) \;=\; \frac{\pi}{8}+\frac{1}{4} \;=\; \frac{2+\pi}{8}.$$

Exercises

1. Compute $\int \sin^3 x \, dx$ using the identity $\sin^2 x = 1 - \cos^2 x$.

2. Evaluate $\int \sin^4 x \cdot \cos^3 x \, dx\,.$

3. Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 \theta \, d\theta\,.$

4. Evaluate $\int \cos^4 x \, dx\,.$

   Hint: Write $\cos^4 x = (\cos^2 x)^2$ and use the first equation in (1).

5. Derive the formula $\int \sec x \, dx = \ln |\sec x + \tan x| + c\,.$

   Hint: Write $\sec x = \frac{\sec x}{1}$ and multiply top and bottom by $\sec x + \tan x$.