

# MATH 136-03 Calculus 2, Spring 2019

## Section 7.6: Strategies for Integration

### Solutions

**Exercises:** Evaluate each of the following integrals by choosing an appropriate technique(s) of integration.

1.  $\int x\sqrt{x^2+1} dx$

**Answer:** This is easiest to compute using a  $u$ -substitution. Let  $u = x^2 + 1$ . Then  $du = 2x dx$  and we have

$$\int x\sqrt{x^2+1} dx = \frac{1}{2} \int 2x\sqrt{x^2+1} dx = \frac{1}{2} \int u^{1/2} du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + c = \frac{1}{3} (x^2+1)^{3/2} + c.$$

2.  $\int x^3\sqrt{x^2+1} dx$

**Answer:** This can also be computed using a  $u$ -substitution, which is easier than doing trig sub. Let  $u = x^2 + 1$ . Then  $du = 2x dx$  and  $x^2 = u - 1$ . If we split the  $x^3$  term into  $x \cdot x^2$ , we can evaluate the integral using the power rule. We find

$$\begin{aligned} \int x^3\sqrt{x^2+1} dx &= \frac{1}{2} \int 2x \cdot x^2\sqrt{x^2+1} dx \\ &= \frac{1}{2} \int (u-1)u^{1/2} du \\ &= \frac{1}{2} \int u^{3/2} - u^{1/2} du \\ &= \frac{1}{2} \left( \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) du \\ &= \frac{1}{5} (x^2+1)^{5/2} - \frac{1}{3} (x^2+1)^{3/2} + c. \end{aligned}$$

3.  $\int t^2 e^{3t} dt$

**Answer:** This integral can be evaluated using integration by parts twice. First, let  $u = t^2$  and  $dv = e^{3t} dt$ . Then  $du = 2t dt$  and  $v = \frac{1}{3} e^{3t}$ . We find

$$\int t^2 e^{3t} dt = t^2 \cdot \frac{1}{3} e^{3t} - \int \frac{1}{3} e^{3t} \cdot 2t dt = \frac{1}{3} t^2 e^{3t} - \frac{2}{3} \int t e^{3t} dt.$$

The remaining integral can be done using integration by parts again. Let  $u = t$  and  $dv = e^{3t} dt$ .

Then  $du = dt$  and  $v = \frac{1}{3}e^{3t}$ . We compute

$$\begin{aligned}
 \int t^2 e^{3t} dt &= \frac{1}{3}t^2 e^{3t} - \frac{2}{3} \int t e^{3t} dt \\
 &= \frac{1}{3}t^2 e^{3t} - \frac{2}{3} \left( t \cdot \frac{1}{3}e^{3t} - \int \frac{1}{3}e^{3t} dt \right) \\
 &= \frac{1}{3}t^2 e^{3t} - \frac{2}{3} \left( \frac{1}{3}t e^{3t} - \frac{1}{9}e^{3t} + c \right) \\
 &= \frac{1}{3}t^2 e^{3t} - \frac{2}{9}t e^{3t} + \frac{2}{27}e^{3t} + c \\
 &= \frac{1}{27}e^{3t} (9t^2 - 6t + 2) + c.
 \end{aligned}$$

4.  $\int_{-\pi/4}^{\pi/4} \sin^4 \theta d\theta$

**Answer:** To evaluate this integral, first notice that the integrand is even. By symmetry, we have  $\int_{-\pi/4}^{\pi/4} \sin^4 \theta d\theta = 2 \int_0^{\pi/4} \sin^4 \theta d\theta$ . Next we apply the formula  $\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$ . Being careful to FOIL the integrand, we find

$$\int_{-\pi/4}^{\pi/4} \sin^4 \theta d\theta = 2 \int_0^{\pi/4} \left( \frac{1}{2}(1 - \cos(2\theta)) \right)^2 d\theta = \frac{1}{2} \int_0^{\pi/4} 1 - 2\cos(2\theta) + \cos^2(2\theta) d\theta.$$

Next we use the formula  $\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta))$ , except we replace  $\theta$  by  $2\theta$ . We find

$$\begin{aligned}
 \frac{1}{2} \int_0^{\pi/4} 1 - 2\cos(2\theta) + \cos^2(2\theta) d\theta &= \frac{1}{2} \int_0^{\pi/4} 1 - 2\cos(2\theta) + \frac{1}{2}(1 + \cos(4\theta)) d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} \frac{3}{2} - 2\cos(2\theta) + \frac{1}{2}\cos(4\theta) d\theta \\
 &= \frac{1}{2} \left( \frac{3}{2}\theta - \sin(2\theta) + \frac{1}{8}\sin(4\theta) \Big|_0^{\pi/4} \right) \\
 &= \frac{1}{2} \left( \frac{3}{2} \cdot \frac{\pi}{4} - \sin(\pi/2) + \frac{1}{8}\sin(\pi) - 0 \right) \\
 &= \frac{3\pi}{16} - \frac{1}{2}.
 \end{aligned}$$

5.  $\int \frac{x^2}{(9-x^2)^{3/2}} dx$  *Hint:* After making a trig substitution, use the identity  $\tan^2 \theta + 1 = \sec^2 \theta$  to compute the integral.

**Answer:** This problem can be computed using the trig substitution  $x = 3\sin \theta$ . Then  $dx =$

$3 \cos \theta d\theta$  and  $9 - x^2 = 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta$ . We find

$$\begin{aligned} \int \frac{x^2}{(9 - x^2)^{3/2}} dx &= \int \frac{9 \sin^2 \theta}{(9 \cos^2 \theta)^{3/2}} \cdot 3 \cos \theta d\theta \\ &= \int \frac{27 \sin^2 \theta \cos \theta}{27 \cos^3 \theta} d\theta \\ &= \int \tan^2 \theta d\theta \\ &= \int \sec^2 \theta - 1 d\theta \\ &= \tan \theta - \theta + c \\ &= \frac{x}{\sqrt{9 - x^2}} - \sin^{-1} \left( \frac{x}{3} \right) + c. \end{aligned}$$

The final step comes from using right-triangle trig and drawing a right triangle with legs  $x$  and  $\sqrt{9 - x^2}$  and hypotenuse 3.

6.  $\int \frac{11x + 6}{(x^2 + 4)(x - 3)} dx$

**Answer:** This problem uses partial fractions. We write

$$\frac{11x + 6}{(x^2 + 4)(x - 3)} = \frac{A}{x - 3} + \frac{Bx + C}{x^2 + 4}$$

and search for the constants  $A$ ,  $B$ , and  $C$ . Multiplying through on both sides by  $(x - 3)(x^2 + 4)$ , we find

$$11x + 6 = A(x^2 + 4) + (Bx + C)(x - 3).$$

Plugging in  $x = 3$  gives  $39 = 13A$  so that  $A = 3$ . Plugging in  $x = 0$  yields  $6 = 4A - 3C$  so that  $C = 2$ . Finally, plugging in  $x = 1$  gives  $17 = 5A - 2(B + C) = 11 - 2B$  so that  $B = -3$ . Thus we have

$$\begin{aligned} \int \frac{11x + 6}{(x^2 + 4)(x - 3)} dx &= \int \frac{3}{x - 3} + \frac{-3x + 2}{x^2 + 4} dx \\ &= 3 \ln |x - 3| + \int \frac{-3x}{x^2 + 4} dx + \int \frac{2}{x^2 + 4} dx \\ &= 3 \ln |x - 3| - \frac{3}{2} \int \frac{2x}{x^2 + 4} dx + \int \frac{2}{x^2 + 4} dx \\ &= 3 \ln |x - 3| - \frac{3}{2} \ln |x^2 + 4| + \tan^{-1} \left( \frac{x}{2} \right) + c, \end{aligned}$$

where the last two integrals are done by  $u$ -sub ( $u = x^2 + 4$ ) and trig sub ( $x = 2 \tan \theta$ ), respectively.

7.  $\int \sin^{-1}(\sqrt{x}) dx$  *Hint:* Start with the  $u$ -substitution  $u = \sqrt{x}$ . Then just go for it.

**Answer:** Letting  $u = \sqrt{x}$ , we have  $du = \frac{1}{2}x^{-1/2} dx$  and thus  $dx = 2\sqrt{x} du = 2u du$ . Thus,

$$\int \sin^{-1}(\sqrt{x}) dx = 2 \int \sin^{-1}(u) \cdot u du.$$

Now use integration by parts with  $t = \sin^{-1}(u)$  and  $dv = u \, du$ . This gives  $dt = \frac{1}{\sqrt{1-u^2}} \, du$  and  $v = u^2/2$ . Thus,

$$2 \int \sin^{-1}(u) \cdot u \, du = 2 \sin^{-1}(u) \cdot \frac{u^2}{2} - 2 \int \frac{u^2}{2\sqrt{1-u^2}} \, du = u^2 \sin^{-1} u - \int \frac{u^2}{\sqrt{1-u^2}} \, du.$$

The remaining integral can be computed using the trig sub  $u = \sin \theta$ . Then  $du = \cos \theta \, d\theta$  and  $\sqrt{1-u^2} = \sqrt{1-\sin^2 \theta} = \cos \theta$ . We find

$$\begin{aligned} \int \frac{u^2}{\sqrt{1-u^2}} \, du &= \int \frac{\sin^2 \theta}{\cos \theta} \cdot \cos \theta \, d\theta \\ &= \int \sin^2 \theta \, d\theta \\ &= \int \frac{1}{2} (1 - \cos(2\theta)) \, d\theta \\ &= \frac{1}{2} \left( \theta - \frac{1}{2} \sin(2\theta) \right) + c \\ &= \frac{1}{2} (\theta - \sin \theta \cos \theta) + c \\ &= \frac{1}{2} \left( \sin^{-1} u - u \cdot \sqrt{1-u^2} \right) + c. \end{aligned}$$

Finally, putting everything together and returning to the original variable  $x$ , we have

$$\begin{aligned} \int \sin^{-1}(\sqrt{x}) \, dx &= 2 \int \sin^{-1}(u) \cdot u \, du \\ &= u^2 \sin^{-1} u - \int \frac{u^2}{\sqrt{1-u^2}} \, du \\ &= u^2 \sin^{-1} u - \left( \frac{1}{2} \left( \sin^{-1} u - u \cdot \sqrt{1-u^2} \right) \right) + c \\ &= u^2 \sin^{-1} u - \frac{1}{2} \sin^{-1} u + \frac{1}{2} u \cdot \sqrt{1-u^2} + c \\ &= x \sin^{-1}(\sqrt{x}) - \frac{1}{2} \sin^{-1}(\sqrt{x}) + \frac{1}{2} \sqrt{x} \cdot \sqrt{1-x} + c. \end{aligned}$$