MATH 136-03 Calculus 2, Spring 2019

Section 6.2: Setting Up Integrals: Volume and Average Value

In this section we learn how to find the volume of a three-dimensional solid by summing up areas of cross sections. We also learn how to find the average value of a function, an important quantity to compute in many applications.

Average Value

Recall that the average of n numbers $y_1, y_2, \ldots, y_n$ is given by

$$\frac{y_1 + y_2 + \cdots + y_n}{n} = \frac{1}{n}\sum_{j=1}^{n} y_j\,.$$

How do we generalize this formula to find the average value of a function? For example, suppose we know the temperature in a room as a function of time. How do we calculate the average temperature over a particular time interval?

We begin by choosing n points $x_1, x_2, \ldots, x_n$ in an interval $[a, b]$ and computing the average of their function values:

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \frac{1}{n}\sum_{j=1}^{n} f(x_j)\,.$$

If we let $\Delta x = \frac{b-a}{n}$, then we find that $\frac{1}{n} = \frac{\Delta x}{b-a}$. Therefore

$$\text{Average Value} \approx \frac{\Delta x}{b-a}\sum_{j=1}^{n} f(x_j) = \frac{1}{b-a}\sum_{j=1}^{n} f(x_j)\,\Delta x\,.$$

The right-hand side of the previous equation can be interpreted as a Riemann sum! Taking the limit as $n \to \infty$ (which means sampling the function at more and more x-values) leads to an integral definition for the average value of a function $f(x)$ on the interval $[a, b]$:

$$\boxed{\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx\,.} \qquad (1)$$

Example 1: Find the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

Answer: Using the formula above, we compute the average value to be

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}\cdot -\cos x\Big|_0^{\pi} = -\frac{1}{\pi}(\cos\pi - \cos 0) = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}\,.$$

One way to visualize the average value of a function is to interpret it as the average "height" (Figure 1).

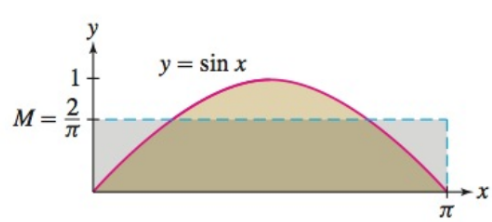


Figure 1: The average value of a function is the average height M that makes the area under the graph equal to the area of the rectangle with height M and base $b - a$.

Exercise 1: Find the average value of $f(x) = 4 - x^2$ on $[-2, 2]$. Illustrate your answer with a graph as in Figure 1.

Exercise 2: Find the average value of $g(t) = \sqrt{t}$ on $[1, 4]$.

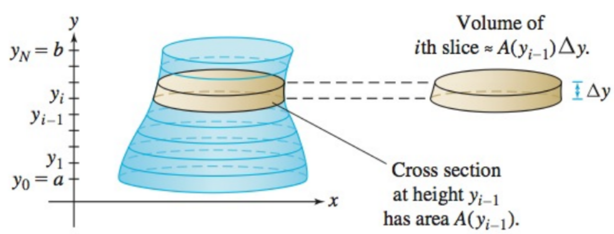
Volume

How do we calculate the volume of a 3D solid? One technique is to divide the solid into thin cross sections and then add up the volumes of each cross section (see Figure 2). Suppose that there are n cross sections each with the same width Δy. If we know the area of the cross section at a height of y_i is given by $A(y_i)$, then the volume of that cross section is $A(y_i)\,\Delta y$ (see Figure 2). Thus, we have

$$\text{Volume} \quad \approx \sum_{i=0}^{n-1} A(y_i)\,\Delta y\,.$$

Once again, this can be interpreted as a Riemann sum. As $\Delta y \to 0$, the sum above becomes an integral over the range of heights in the y-direction:

$$\boxed{\text{Volume} \quad = \int_a^b A(y)\,dy\,.} \qquad (2)$$



DF **FIGURE 2** Divide the solid into thin horizontal slices. Each slice is nearly a right cylinder whose volume can be approximated as area times height.

Note that there is nothing special here about taking horizontal cross sections. This technique for finding the volume of a solid works perfectly well using vertical cross sections (perpendicular to the x-axis) between $x = a$ and $x = b$. In this case formula (2) becomes

$$\boxed{\text{Volume} \quad = \int_a^b A(x)\,dx\,.} \qquad (3)$$

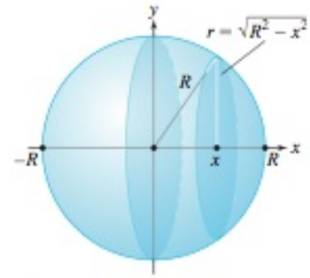


Figure 3: Finding the volume of a sphere of radius R using circular cross sections perpendicular to the x-axis.

Exercise 3: Use Figure 3 and circular cross sections perpendicular to the x-axis to show that the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$.

Exercise 4: Find the volume of the solid whose base is the unit circle $x^2 + y^2 = 1$ with cross sections perpendicular to the x-axis equal to triangles whose height and base are equal.

Hint: Find the base of each triangle as a function of x.