MATH 135-08, 135-09 Calculus 1, Fall 2017

Worksheet for Sections 3.5 and 3.6

Higher Derivatives and Trig Functions

3.5 Higher Derivatives

Recall that the derivative $f'(x)$ is the **slope function**: it gives the slope of the function f at the point x. Since $f'(x)$ is itself a function, we can ask, "What is the derivative of the derivative?" This is called the **second derivative.**

Definition 0.1 $\frac{d}{dx}(f'(x)) = f''(x)$ *is called the* <u>*second derivative*</u>*. It measures the rate of change of the slope function* $f'(x)$*.*

To find the second derivative of a function f, we simply take the derivative twice. For example, if $f(x) = 3x^2 - 5x$, then $f'(x) = 6x - 5$ and $f''(x) = 6$, applying the Power Rule twice. Note that in this particular case, f is a quadratic function, f' is a linear function and f'' is a constant function. Each time we take the derivative, the power decreases by one.

Exercise 0.2 *Suppose that* $f(x) = 2x^4 - 3x + e^x$*. Find* $f'(x)$ *and* $f''(x)$*.*

Leibniz Notation for the Second Derivative

To write the second derivative using Leibniz notation, we use

$$f''(x) = \frac{d^2 f}{dx^2} \quad \text{or} \quad \text{if } y = f(x), \text{ then } f''(x) = \frac{d^2 y}{dx^2}.$$

Concavity

If f' tells us the slope of the function, what does f'' represent? To answer this question, note that if $f'(x) > 0$, then the slope is positive at x and the function is **increasing** there (moving upwards from left to right). On the other hand, if $f'(x) < 0$, then the slope is negative at x and the function is **decreasing** there (moving downwards from left to right).

$$\boxed{\begin{array}{l} f' > 0 \implies f \text{ is increasing} \\ f' < 0 \implies f \text{ is decreasing} \end{array}} \tag{1}$$

Next, suppose that $f''(x) > 0$. This means that $(f')' > 0$, so the slopes of f are increasing. There are two possibilities. If the slopes are positive and getting bigger, then the curve is getting steeper, increasing at a faster and faster rate (like e^x). On the other hand, the slopes could be negative but getting closer to 0 (e.g., $m = -3, m = -1.4, m = -0.4$). In this case, the curve is decreasing but beginning to flatten out (such as e^{-x}). In either case, we say that the graph of f is **concave up**.

Similarly, if $f''(x) < 0$, then the slopes are decreasing. There are two cases here as well. Either the slopes are positive and getting smaller (like $\ln x$) or they are negative and getting more negative (such as $\ln(-x)$ for $x < 0$). In the first case, the curve is increasing, but starting to flatten out, while in the second case, the curve is decreasing and becoming more and more steep. In either of these cases, we say that the graph of f is **concave down**. In sum, we have

$$\boxed{\begin{array}{r} f'' > 0 \implies f \text{ is concave up} \\ f'' < 0 \implies f \text{ is concave down} \end{array}} \tag{2}$$

Exercise 0.3 *Sketch the graph of a function g such that $g'(x) < 0$ and $g''(x) > 0$ everywhere along the function.*

Exercise 0.4 *Former President Nixon once famously said, "Although the rate of inflation is increasing, it is increasing at a decreasing rate." (He was trying to assuage the fears of a nervous US public about the rapidly rising cost of goods.) Let $r(t)$ = the rate of inflation. What are the signs ($+, -$ or 0) of $r'(t)$ and $r''(t)$?*

Higher Derivatives

Just as we can take the derivative of the derivative, we can also take the derivative of the second derivative, called the **third derivative**. Thus, the third derivative of f, denoted as $f'''(x)$, means to take the derivative of f three times. It represents the change in the concavity of a function.

But there is no reason to stop at just three derivatives. The fourth derivative, denoted $f^{iv}(x)$, is the derivative of $f'''(x)$ and represents the concavity of the second derivative or reveals the slopes of f'''. Although this might appear to be a twisted mathematical extension of the derivative, higher derivatives frequently arise in applications. For instance, if $s(t)$ represents the position of a particle, then the third derivative $s'''(t)$ is known as the **jerk** (from physics). The fourth derivative shows up in differential equations that model the flow of waves or turbulence around an airplane.

Exercise 0.5 *Suppose that $f(x) = xe^x$. Find and simplify $f'(x), f''(x)$ and $f'''(x)$. In general, what is $f^{(n)}(x)$, the nth derivative of f. Try and find a formula that involves n.*

3.6 Trigonometric Functions

The following key derivatives can be proven using the definition of the derivative and the trig identities $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$ and $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$.

Theorem 0.6 *Assume that x is measured in radians.*

$$\boxed{\frac{d}{dx}(\sin x) \;=\; \cos x \qquad \textit{and} \qquad \frac{d}{dx}(\cos x) \;=\; -\sin x\,.} \tag{3}$$

To prove the first formula, let $f(x) = \sin x$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \cdot \sin h}{h} \\
&= \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \;=\; \cos x,
\end{aligned}$$

where we have made use of the two important trig limits $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$. A similar proof can be used to show that the derivative of $\cos x$ is $-\sin x$.

The derivatives of the other four trig functions can now be found using the quotient rule. For instance, we have that $\frac{d}{dx}(\tan x) \;=\; \sec^2 x$ because

$$\frac{d}{dx}(\tan x) \;=\; \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \;=\; \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \;=\; \frac{1}{\cos^2 x} \;=\; \left(\frac{1}{\cos x}\right)^2 \;=\; \sec^2 x.$$

Exercise 0.7 *Use the quotient rule to show that $\frac{d}{dx}(\cot x) \;=\; -\csc^2 x$.*

Exercise 0.8 *Show that* $\dfrac{d}{dx}(\sec x) = \sec x \cdot \tan x$.

Exercise 0.9 *If* $g(x) = x^3 \sin x$, *find and simplify* $g'(x)$ *and* $g''(x)$.

Exercise 0.10 *Find the equation of the tangent line to* $y = \dfrac{1-3\theta}{\cos\theta + \theta\sin\theta}$ *at* $\theta = 0$.