MATH 135-08, 135-09 Calculus 1, Fall 2017

Limits and Continuity: Worksheet for Section 2.4

Continuity

Intuitively, a continuous function is one that can be drawn without having to lift up your pencil. Functions that are *not* continuous have holes, jumps, asymptotes, or places where limits don't exist (e.g., infinite oscillations). The precise mathematical definition involves limits.

Definition 0.1 *A function $f(x)$ is continuous at $x = a$ if*

$$\boxed{\lim_{x\to a} f(x) \;=\; f(a).} \tag{1}$$

For a function to be continuous at the point $x = a$, there are three conditions:

1. $f(a)$ must exist (there must be a function value).
2. The limit of the function as x approaches a must exist, and it must equal a real number (so ∞ or $-\infty$ is not ok).
3. The limit must equal the function value.

Types of Discontinuities

- If the left-hand and right-hand limits at $x = a$ each exist, but are not equal to each other, then f has a **jump discontinuity** at $x = a$.
- If $\lim_{x\to a} f(x)$ exists, but does not equal the function value $f(a)$, then f has a **removable discontinuity** at $x = a$. If we redefine $f(a)$ to be the value of the limit, then the function becomes continuous at $x = a$, and we have "removed" the discontinuity.
- If either of the one sided limits go to ∞ or $-\infty$, then f has an **infinite discontinuity** at $x = a$.

One-Sided Continuity

A function is **left continuous** at $x = a$ if the left-hand limit exists and equals the function value $f(a)$. Similarly, it is **right continuous** at $x = a$ if the right-hand limit exists and equals the function value $f(a)$. In other words,

- The function $f(x)$ is left continuous at $x = a$ if $\lim_{x\to a^-} f(x) = f(a)$.
- The function $f(x)$ is right continuous at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$.
- The function $f(x)$ is continuous at $x = a$ if and only if it is *both* left and right continuous at $x = a$.

Exercise 0.2 *Consider the piecewise function $f(x)$ defined as follows:*

$$f(x) \;=\; \begin{cases} 6 & \text{if } x \le 1 \\ 4 - x & \text{if } 1 < x \le 4 \\ (x-4)^2 & \text{if } x > 4. \end{cases}$$

(i) *Carefully sketch the graph of $f(x)$.*

(ii) *Describe the type of continuity (left, right, neither, continuous) at $x = 1$.*

(iii) *Describe the type of continuity (left, right, neither, continuous) at $x = 4$.*

(iv) *How would you change the definition of the function over $1 < x \le 4$ (the middle portion) to make it continuous for all real numbers?*

Note: Polynomials, rational functions, exponentials, logs, and trig functions are all continuous on their domains. Moreover, the composition of continuous functions is also continuous. For example, the function

$$g(x) \;=\; \sin\left(e^{x^2-1}\right)$$

is continuous since it is the composition of the continuous functions $x^2 - 1$, e^x and $\sin x$. To find limits of continuous functions, we simply evaluate the function at the point in question (i.e, just plug it in!) Thus, since g is continuous at $x = 1$, we have

$$\lim_{x \to 1} g(x) \;=\; g(1) \;=\; \sin(e^{1-1}) \;=\; \sin(1).$$

Exercise 0.3 *Use continuity to find the value of each limit.*

(i) $\lim_{t \to 3} \log_5 \left[\cos(t-3) + 4\right]$

(ii) $\lim_{x \to -1} \dfrac{3^x}{\sqrt{x+5}}$.