MATH 135 Calculus 1, Spring 2016

Worksheet for Sections 1.2 and 1.3

1.2 Linear and Quadratic Functions

Linear Functions

A **linear function** is one of the form $f(x) = mx + b$, where m and b are arbitrary constants. It is "linear" in x (no exponents, fractions, trig, etc.). The graph of a linear function is a line. The constant m is the **slope** of the line and this number has the same value **everywhere** on the line. If (x_1, y_1) and (x_2, y_2) are any two points on the line, then the slope is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \qquad (\Delta = \text{"Delta," means change}).$$

This number will be the same no matter which two points on the line are chosen.

Two important equations for a line are:

1. Slope-intercept form: $y = mx + b$ (m is the slope and b is the y-intercept.)
2. Point-slope form: $y - y_0 = m(x - x_0)$ (m is the slope and (x_0, y_0) is any point on the line.)

If $m > 0$, then the line is increasing while if $m < 0$, the line is decreasing. When $m = 0$, the line has zero slope and is horizontal (a constant function). Two lines are **parallel** when they have the same slope, while two lines are **perpendicular** if the product of their slopes is -1.

Exercise 0.1 *Find the equation of the line with the given information.*

(a) *The line passing through the points* $(-2, 3)$ *and* $(4, 1)$.

(b) *The line parallel to* $2y + 5x = 0$, *passing through* $(2, 5)$.

(c) *The line perpendicular to* $2y + 5x = 0$, *passing through* $(2, 5)$.

Quadratic Functions

A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are arbitrary constants and $a \neq 0$. The graph of a quadratic function is a **parabola**, an important curve that arises in many fields (e.g., physics, acoustics, astronomy). The parabola opens up when $a > 0$ and down for $a < 0$. The x-coordinate of the vertex of the parabola is located at $x = -b/(2a)$.

A quadratic function may have either two, one, or zero real roots, found by solving the equation $ax^2 + bx + c = 0$. The number of roots is determined by the **discriminant** $D = b^2 - 4ac$. If $D > 0$, then there are two distinct real roots. If $D = 0$, then there is one root, called a **repeated root**. If $D < 0$, then there are no real roots. The roots can be found either by factoring $ax^2 + bx + c = 0$ (in special cases) or by using the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}.$$

Note the appearance of D under the square root. If $D < 0$, then the roots are not real. In this case, the parabola does not cross the x-axis. If $D > 0$, then there are two real roots and these are equidistant from the x-coordinate of the vertex of the parabola.

Exercise 0.2 *Find the roots of the quadratic function $f(x) = -3x^2 + 9x + 12$ in two different ways:* **(a)** *by factoring, and* **(b)** *by using the quadratic formula. Use this information to sketch a graph of $f(x)$.*

Completing the Square

One important algebraic technique for understanding quadratic functions is **completing the square**. This means to write the function as a multiple of a perfect square plus a constant. Here is an example:

$$\begin{aligned} 2x^2 + 6x + 7 &= 2(x^2 + 3x + \underline{\qquad\quad}) + 7 \\ &= 2\left(x^2 + 3x + \frac{9}{4}\right) + 7 - \frac{9}{2} \\ &= 2\left(x + \frac{3}{2}\right)^2 + \frac{5}{2} \end{aligned}$$

The key to completing the square is to add and subtract the correct constant (9/4 in the above example) to make the factorization into a perfect square. After factoring out the leading coefficient (on the x^2 term), the missing constant is found by cutting the coefficient of the linear term in half, and then squaring.

Exercise 0.3 *Complete the square for the function* $f(x) = x^2 - 7x + 15$. *In other words, write it in the form* $f(x) = (x - h)^2 + k$. *(You have to figure out what the constants* h *and* k *are.)*

Exercise 0.4 *By completing the square, find the range of the quadratic function* $f(x) = 4x^2 - 24x + 31$. *What are the coordinates of the vertex of the corresponding parabola?*

Exercise 0.5 (Challenge Problem) *Find the quadratic function that is even and passes through the points* $(-1, 1)$ *and* $(2, 13)$.

1.3 The Basic Classes of Functions

What follows is a brief catalog of the standard functions that we will be studying this semester. It is important to understand the properties of each function: defining equation, typical graph, domain and range, when it is used, etc.

Linear: $L(x) = mx + b$ Examples: $L(x) = 3x - 1,\ L(x) = -2x + \sqrt{3},\ L(x) = -7.$

Linear functions have a **constant** rate of change (determined by the slope m). The graph of a linear function is a line. It moves upwards from left to right if $m > 0$, downwards from left to right if $m < 0$, and is horizontal when $m = 0$. Remember that a vertical line $x = c$ is **not** a function! Linear functions are often used as a first approximation to a graph. This is called the **tangent line**, the primary focus of Calc 1.

Polynomial: $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_n \neq 0$ Example: $p(x) = 3x^5 - 2x^2 + 14.$

The constant a_n is called the **leading coefficient** and n is the **degree** of the polynomial. In general, an nth degree polynomial has n roots (zeros), although some of these may be complex (non-real). If $n = 2$, the polynomial is a **quadratic** function; if $n = 3$, it is called a **cubic**; if $n = 4$, it is called a **quartic**, etc. The domain of a polynomial function is $\mathbb{R}$, the set of all real numbers. Typically, the graph of an nth degree polynomial has $n - 1$ humps (facing up or down). Polynomials are often used to approximate more complicated functions. They are particularly nice b/c the derivative and integral are easy to calculate using the power rule (to be discussed later).

Rational: $R(x) = \dfrac{p(x)}{q(x)}$ Example: $R(x) = \dfrac{2x^3 - 5x^2 + 12}{x^2 - 2x - 3}$

A rational function is the **ratio** of two polynomials. The domain of a rational function is all real numbers except for the roots of $q(x)$, since a root of the denominator would make the function undefined. Typically, $R(x)$ has a **vertical asymptote** at the x-values which are roots of $q(x)$. A vertical asymptote is a dashed vertical line which the graph of the function approaches, either upwards (toward $+\infty$) or downwards (toward $-\infty$).

Exercise 0.6 *Find the domain of the function*

$$R(x) \ = \ \frac{3x^4 - 7x^3 + \pi}{x^2 - 16}.$$

Exponential: $f(x) = b^x$, where b is some positive constant. Examples: $2^x, (1/2)^x, e^x, 1.003^x$.

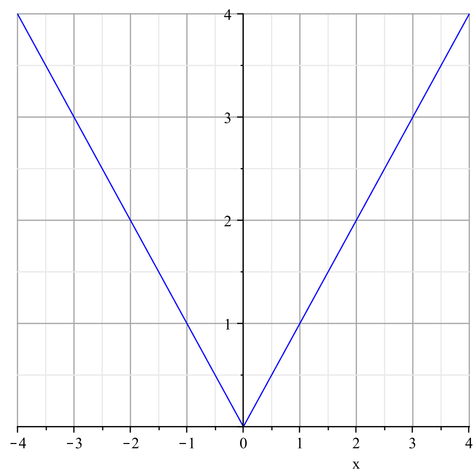
Exponential functions are very important in fields such as economics, population biology, physics, mathematical modeling, and finance, to name a few. Any quantity that grows or decays based on how much of that quantity is present is described by an exponential function.

Note: The variable in an exponential function is an **exponent**. There is a huge difference between x^2 (squaring function) and 2^x (doubling function). Exponential functions grow very, very fast. Their domains are all real numbers. The base of an exponential function $f(x) = b^x$ is the constant b, which is always assumed to be positive. If $b > 1$, we have exponential **growth**, while if $b < 1$, we have exponential **decay.** We will discuss these important functions further in Section 1.6.

Piecewise: A function that has multiple parts defined on different domains.

Sometimes a function is split into separate pieces, with a different definition used on each domain. The most familiar example is the V graph of the absolute value function (see figure below):

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$



We graph the line $y = x$ (positive slope) over the domain $x \geq 0$ (the right-hand side of the graph). Then we graph the line $y = -x$ (negative slope) over the domain $x < 0$ (the left-hand side of the graph). Since both lines meet at the point $(0, 0)$, we obtain a V-shaped graph with the vertex at $(0, 0)$.

Exercise 0.7 *Carefully draw the graph of*

$$g(x) = \begin{cases} (x-3)^2 & \textit{if } x \geq 3 \\ 2x - 3 & \textit{if } -1 < x < 3 \\ -5 & \textit{if } x \leq -1\,. \end{cases}$$

Composing Functions: Example: $f(x) = 2^x, g(x) = -3x + 1$ yields $f(g(x)) = 2^{-3x+1}$.

One way to create a new function from two functions is to compose them together. The notation for composition of functions is $f \circ g$ which means the function $f(g(x))$, pronounced "f of g of x." In this case, x is first plugged into the function g, and then the output $g(x)$ is plugged into f. For example, suppose that we define the function $h(x) = f(g(x))$. If $g(2) = 7$, and $f(7) = -3$, then $h(2) = -3$ because

$$h(2) \;=\; f(g(2)) \;=\; f(7) \;=\; -3.$$

If we flip the order of f and g, we usually obtain a new function, that is, $f(g(x)$ and $g(f(x))$ are **different** functions. The domain of the function $f \circ g$ is all x in the domain of g that map into the domain of f.

Exercise 0.8 *Suppose that* $f(x) = \sqrt{x}$ *and* $g(x) = 3x + 1$. *Find* $f(g(x))$ *and* $g(f(x))$ *and their respective domains.*

Exercise 0.9 (Challenge Problem) *Suppose that* $f(x) = 5x - 3$. *Find a function* $g(x)$ *such that* $g(f(x)) = 2x$.