MATH 134 Calculus 2 with FUNdamentals

SOLUTIONS to Practice Exam #3

1. **Quickies/Multiple Choice:**

(a) Given the infinite series $\sum_{n=1}^{\infty} a_n$, suppose that $\lim_{n\to\infty} a_n = 0$. What can we definitively conclude about the series?

(i) The series converges.
(ii) The series converges and the sum is 0.
(iii) The series diverges.
(iv) The series may or may not converge; the test is inconclusive.

Answer: **(iv)** If $\lim_{n\to\infty} a_n = 0$, the series may or may not converge. A different test is needed to determine convergence. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ each satisfy $\lim_{n\to\infty} a_n = 0$. However, the first series diverges (Harmonic Series) while the second converges (p-series with $p = 2$).

On the other hand, if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. This is the nth term test.

(b) Determine whether the sequence below converges or diverges. If it converges, state the limit.

$$a_n = \tan^{-1}\left(\frac{n^2}{2n+3}\right)$$

Answer: The sequence converges to $\pi/2$. This follows by first evaluating the limit inside the parentheses. Since the numerator is a higher power than the denominator, the limit inside goes to ∞ (one could also use L'Hôpital's Rule). Since $\lim_{x\to\infty} \tan^{-1} x = \pi/2$, the sequence converges to $\pi/2$.

(c) Find the sum of the infinite series $16 - 4 + 1 - \frac{1}{4} + \frac{1}{16} - + \cdots$.

Answer: This is a geometric series with ratio $r = -1/4$. The ratio can be found by inspection or by dividing any term by the term proceeding it (e.g., $-4 \div 16 = -1/4$, or $1 \div (-4) = -1/4$). Then, using the formula for the sum of a geometric series, we have

$$S = \frac{a}{1-r} = \frac{16}{1-(-1/4)} = \frac{16}{\frac{5}{4}} = \frac{64}{5} \text{ or } 12.8.$$

2. **Compound Interest and Present Value:** (round answers to the nearest cent)

   (a) Suppose that \$40,000 is invested in an account paying interest at an annual rate of 6% with the interest compounded **monthly.** How much money will be in the account after 10 years?

   Answer: Use the formula $P(t) = P_0 \left(1 + \frac{r}{M}\right)^{Mt}$ with $P_0 = 40{,}000, r = 0.06, M = 12$, and $t = 10$. The amount of money in the account after 10 years is

$$P(10) \;=\; 40{,}000 \left(1 + \frac{0.06}{12}\right)^{12 \cdot 10} \;\approx\; \$72{,}775.87\,.$$

   (b) Congratulations, you just won a million dollars in the lottery! However, you do not get all of your money now; you will receive four yearly payments of \$250,000 beginning immediately. Assuming an interest rate of 4%, what is the present value of your prize? How much do you "lose" by not receiving the full prize today?

   Answer: We need to compute the present value of each yearly payment and then add them together to see how they compare with one million. The first payment starts the clock (time $t = 0$). We obtain

$$\begin{aligned} 250{,}000 + 250{,}000e^{-0.04 \cdot 1} + 250{,}000e^{-0.04 \cdot 2} + 250{,}000e^{-0.04 \cdot 3} &= \\ 250{,}000\left(1 + e^{-0.04} + e^{-0.08} + e^{-0.12}\right) &\approx \$942{,}706.56\,. \end{aligned}$$

   The "loss" on our winnings is one million minus the present value or \$57,293.44 .

3. **Improper Integrals:** Determine whether each improper integral converges or diverges. If the integral converges, give the exact value of the integral.

   (a) $\displaystyle\int_1^{\infty} e^{-3x}\,dx$

   Answer: The integral converges to $\frac{1}{3e^3}$. We have

$$\begin{aligned} \int_1^{\infty} e^{-3x}\,dx &= \lim_{b\to\infty} \int_1^{b} e^{-3x}\,dx \\ &= \lim_{b\to\infty} -\frac{1}{3}e^{-3x}\Big|_1^b \\ &= \lim_{b\to\infty} -\frac{1}{3}e^{-3b} - \left(-\frac{1}{3}e^{-3}\right) \\ &= \lim_{b\to\infty} -\frac{1}{3e^{3b}} + \frac{1}{3}e^{-3} \\ &= 0 + \frac{1}{3}e^{-3} \\ &= \frac{1}{3e^3}\,. \end{aligned}$$

(b) $\displaystyle\int_2^5 \frac{1}{2x-4}\,dx$

Answer: The integral diverges. It can be computed using a u-substitution with $u = 2x-4$ and $du = 2\,dx$ or $dx = \frac{1}{2}\,du$. Notice that the "bad" point is $x = 2$ since this value of x makes the denominator 0 (vertical asymptote). We have

$$\begin{aligned}
\int_2^5 \frac{1}{2x-4}\,dx &= \lim_{b\to 2^+} \int_b^5 \frac{1}{2x-4}\,dx \\
&= \lim_{b\to 2^+} \frac{1}{2}\int_{2b-4}^{6} \frac{1}{u}\,du \qquad (u = 2x-4, du = 2\,dx) \\
&= \lim_{b\to 2^+} \frac{1}{2}\ln|u|\Big|_{2b-4}^{6} \\
&= \lim_{b\to 2^+} \frac{1}{2}\ln 6 - \frac{1}{2}\ln|2b-4| \\
&= \frac{1}{2}\ln 6 + \infty \quad \text{(diverges)},
\end{aligned}$$

since $\displaystyle\lim_{x\to 0^+} \ln x = -\infty$.

4. **Probability:** Consider the piecewise function

$$p(x) = \begin{cases} 0 & \text{if } x < 0 \\ \dfrac{C}{(x+2)^3} & \text{if } x \ge 0 \end{cases}$$

(a) Find the value of C that makes $p(x)$ a probability density function.

Answer: $C = 8$. In order for $p(x)$ to be a probability density function, it must satisfy $\displaystyle\int_{-\infty}^{\infty} p(x)\,dx = 1$. Given the fact that $p(x) = 0$ for $x < 0$, this condition is equivalent to

$$\int_0^\infty \frac{C}{(x+2)^3}\,dx = 1\,.$$

We have

$$\begin{aligned}
\lim_{b\to\infty} \int_0^b \frac{C}{(x+2)^3}\,dx = 1 &\implies \lim_{b\to\infty} \int Cu^{-3}\,dx = 1 \qquad (u = x+2, du = dx) \\
&\implies \lim_{b\to\infty} \frac{C}{-2}(x+2)^{-2}\Big|_0^b = 1 \\
&\implies -\frac{C}{2}\lim_{b\to\infty}\left(\frac{1}{(b+2)^2} - \frac{1}{2^2}\right) = 1 \\
&\implies -\frac{C}{2}(0 - 1/4) = 1 \\
&\implies \frac{C}{8} = 1 \text{ or } C = 8\,.
\end{aligned}$$

(b) Using your value of C from part **(a)**, find $P(0 \le x \le 2)$.

Answer: $3/4$. We have

$$P(0 \le x \le 2) \;=\; \int_0^2 \frac{8}{(x+2)^3}\,dx \;=\; 8 \cdot -\frac{1}{2}(x+2)^{-2}\Big|_0^2 \;=\; -\frac{4}{(x+2)^2}\Big|_0^2 \;=\; -\frac{1}{4} - (-1) \;=\; \frac{3}{4}\,.$$

(c) Using your value of C from part **(a)**, find $P(x \ge 2)$.

Answer: $1/4$. The easy way is to realize that

$$P(x \ge 2) \;=\; P(x \ge 0) - P(0 \le x \le 2) \;=\; 1 - \frac{3}{4} \;=\; \frac{1}{4}\,.$$

Computing the improper integral yields the same result.

5. **Infinite Series:** Determine whether the given infinite series converges or diverges using any of the tests discussed in class. You must state the test used and provide valid reasons to receive full credit.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4+2}$

Answer: This series diverges by the integral test. The integral can be evaluated using a u-sub with $u = x^4 + 2$ and $du = 4x^3\,dx$ or $dx\,x^3 = \frac{1}{4}\,du$. We have

$$\begin{aligned}
\int_1^{\infty} \frac{x^3}{x^4+2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{x^3}{x^4+2}\,dx \\
&= \lim_{b\to\infty} \int_1^b \frac{1}{u}\cdot\frac{1}{4}\,du \\
&= \lim_{b\to\infty} \frac{1}{4}\ln|x^4+2|\Big|_1^b \\
&= \lim_{b\to\infty} \frac{1}{4}\big(\ln(b^4+2) - \ln 3\big) \\
&= \infty\,,
\end{aligned}$$

since $\lim_{x\to\infty} \ln x = \infty$.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\sqrt{n}\,)^3}$

Answer: This series converges by the p-series test. Since $(\sqrt{n}\,)^3 = (n^{1/2})^3 = n^{3/2}$, the series is equivalent to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$, which is a p-series with $p = 3/2$. Since $3/2 > 1$, the series converges by the p-series test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

Answer: This series diverges by the nth term test. Since $\frac{n}{\sqrt{n^2+1}} \approx \frac{n}{\sqrt{n^2}} = 1$,

$$\lim_{n\to\infty} a_n \;=\; 1 \;\neq\; 0$$

and the series diverges by the nth term test.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^n$

Answer: This series converges because it is a geometric series with ratio $r = -2/5$. The first few terms of the series are

$$\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^n \;=\; -\frac{2}{5} + \frac{4}{25} - \frac{8}{125} + - \cdots\,,$$

which is a geometric series with ratio $r = -2/5$. Since $|-2/5| = 2/5 < 1$, the series converges.

6. **Calculus Potpourri:**

(a) Find the arc length of $y = 2x^{3/2}$ from $x = 0$ to $x = 1/3$.

Answer: 14/27. Using the formula $L = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx$, we have

$$\frac{dy}{dx} \;=\; \frac{3}{2}\cdot 2x^{1/2} \;=\; 3x^{1/2} \quad\Longrightarrow\quad \left(\frac{dy}{dx}\right)^2 \;=\; (3\sqrt{x}\,)^2 \;=\; 9x.$$

The ensuing integral can be done by a u-substitution with $u = 1 + 9x$. Then $du = 9\,dx$ or $dx = \frac{1}{9}\,du$. Also, $x = 0$ implies $u = 1$ while $x = 1/3$ implies $u = 4$. We have

$$\begin{aligned}
\int_0^{1/3} \sqrt{1+9x}\, dx \;&=\; \frac{1}{9}\int_1^4 \sqrt{u}\, du \\
&=\; \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big|_1^4 \\
&=\; \frac{2}{27}\left(4^{3/2} - 1^{3/2}\right) \\
&=\; \frac{2}{27}\,(8-1) \\
&=\; \frac{14}{27}\,.
\end{aligned}$$

(b) Suppose that the demand curve for a certain commodity is given by $p(x) = 60 - 4x$ and the supply curve is given by $s(x) = \frac{1}{10}x^2 + 10$. Find the equilibrium price $\overline{p}$ and then compute the consumer and producer surplus at the equilibrium price.

Answer: The equilibrium price is $\overline{p} = \$20$, the consumer surplus is \$200, and the producer surplus is $200/3 \approx \$66.67$.

To find the equilibrium price, we find where $p(x)$ and $s(x)$ intersect. Solving $p(x) = s(x)$ leads to the quadratic equation $\frac{1}{10}x^2 + 4x - 50 = 0$ or $x^2 + 40x - 500 = 0$. This factors as $(x + 50)(x - 10) = 0$, which means $\overline{x} = 10$. Plugging $x = 10$ into either $p(x)$ or $s(x)$ gives the equilibrium price of $\overline{p} = \$20$.

Next, using the integral formula for consumer surplus, we find that

$$\text{CS} = \int_0^{10} (60 - 4x) - 20 \, dx = \int_0^{10} 40 - 4x \, dx = 40x - 2x^2\Big|_0^{10} = \$200\,.$$

Using the integral formula for producer surplus, we find that

$$\text{PS} = \int_0^{10} 20 - \left(\frac{1}{10}x^2 + 10\right) \, dx = \int_0^{10} 10 - \frac{1}{10}x^2 \, dx = 10x - \frac{1}{30}x^3\Bigg|_0^{10} = \frac{200}{3} \approx \$66.67\,.$$

(c) Find the total area of the infinitely many circles on the interval $[0, 1]$ in the figure below.

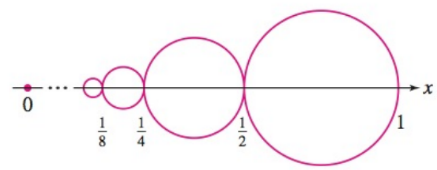

FIGURE 1

Answer: $\pi/12$. The area of the biggest circle is $\pi(\frac{1}{4})^2 = \pi/16$. The area of the next biggest circle is $\pi(\frac{1}{8})^2 = \pi/64$, and the area of the third largest circle is $\pi(\frac{1}{16})^2 = \pi/256$. Each successive circle has half the radius of the previous one and thus $1/4$ the area. The total area is given by the sum of the infinite geometric series

$$\frac{\pi}{16} + \frac{\pi}{64} + \frac{\pi}{256} + \cdots + \frac{\pi}{4^n} + \cdots$$

Since the first term is $\pi/16$ and the ratio is $r = 1/4$, the sum is

$$S = \frac{a}{1 - r} = \frac{\pi/16}{1 - 1/4} = \frac{\frac{\pi}{16}}{\frac{3}{4}} = \frac{\pi}{16} \cdot \frac{4}{3} = \frac{\pi}{12}\,.$$