MATH 134 Calculus 2 with FUNdamentals

Practice Exam #1 SOLUTIONS

1. Let $g(x) = \ln x$ over the interval $1 \leq x \leq 3$.

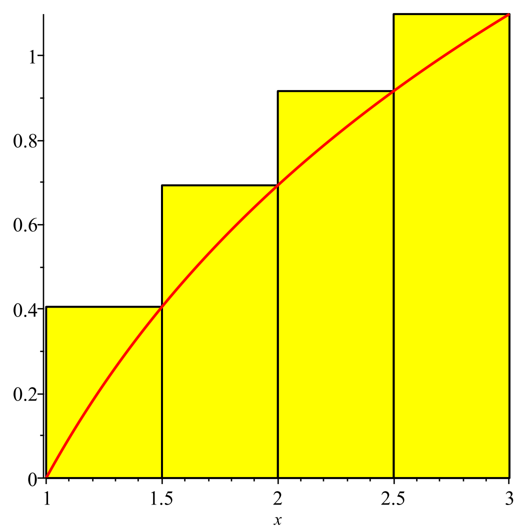
(a) Approximate the area (to three decimal places) under the graph of $g(x) = \ln x$ from $1 \leq x \leq 3$ by using **four** equal subintervals and right endpoints (i.e., calculate the right-hand sum R_4).

Answer: The width of each rectangle is $\Delta x = (3-1)/4 = 1/2$. Evaluating g at the right endpoints of each subinterval gives an estimated area of

$$R_4 = \frac{1}{2}\left(g(1.5) + g(2) + g(2.5) + g(3)\right) = \frac{1}{2}(\ln 1.5 + \ln 2 + \ln 2.5 + \ln 3) \approx 1.557\,.$$

(b) Sketch a graph of $g(x)$ over $[1,3]$ and draw the four rectangles used to compute R_4. Based on your figure, is your estimate in part **(a)** an underestimate, an overestimate, or can this not be determined?

Answer: The value in part **(a)** is an overestimate because g is an increasing function (see the figure below).

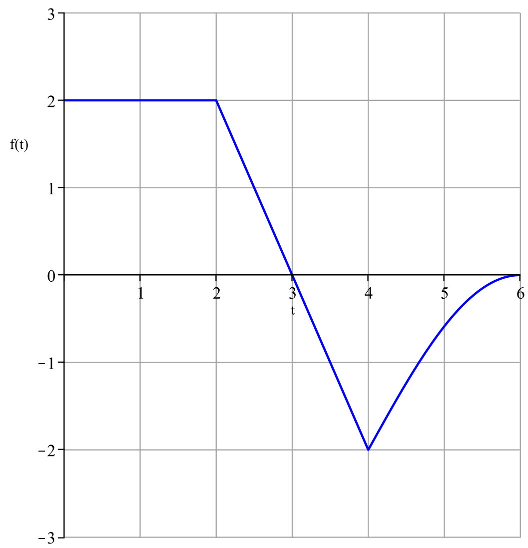


(c) Approximate the area (to three decimal places) under the graph of $g(x) = \ln x$ from $1 \leq x \leq 3$ by using **four** equal subintervals and midpoints (i.e., calculate the midpoint sum M_4).

Answer: The width of each rectangle is still $1/2$. Evaluating g at the midpoints of each subinterval gives an estimated area of

$$\begin{aligned} M_4 &= \frac{1}{2}\left(g(1.25) + g(1.75) + g(2.25) + g(2.75)\right) \\ &= \frac{1}{2}(\ln 1.25 + \ln 1.75 + \ln 2.25 + \ln 2.75) \\ &\approx 1.303\,. \end{aligned}$$

2. Define $A(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 6$, where the graph of $f(t)$ is shown below.



(a) Find $A(0)$ and $A(3)$.

Answer: Find the area under the curve. $A(0) = 0$ and $A(3) = 2 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$ (a rectangle plus a triangle).

(b) Find $A'(2), A'(3), A''(2)$, and $A''(3)$, if they exist.

Answer: Using FTC, part 2, we have that $A'(x) = f(x)$ and thus $A'(2) = f(2) = 2$ and $A'(3) = f(3) = 0$, as can be seen from the graph (find the height).

Next, $A'(x) = f(x)$ implies that $A''(x) = f'(x)$ (differentiate both sides with respect to x). This means that $A''(2) = f'(2)$ does not exist because there is a corner at $t = 2$ and $A''(3) = f'(3) = -2$ (slope of the line at $t = 3$.)

(c) On what interval(s) is $A(x)$ increasing?

Answer: Since $A'(x) = f(x)$ and since a function is increasing whenever its derivative is positive, we see that A is increasing whenever $f > 0$, or when $0 < x < 3$.

(d) On what interval(s) is $A(x)$ concave up?

Answer: Since $A''(x) = f'(x)$ and since a function is concave up whenever its second derivative is positive, we see that A is concave up whenever $f' > 0$ (so f is increasing), or when $4 < x < 6$.

3. Evaluate each of the following integrals, giving the **exact** answer (no decimals) for parts **(e)** and **(f)**.

(a) $\int 10x^4 + \sqrt{x} - \pi\,dx$

Computing each antiderivative separately, we obtain

$$10 \cdot \frac{1}{5}x^5 + \frac{2}{3}x^{3/2} - \pi x + c \;=\; 2x^5 + \frac{2}{3}x^{3/2} - \pi x + c,$$

using the power rule.

(b) $\displaystyle\int 3^x + \sin(4x) - \frac{2}{x}\, dx$

Answer: Computing each antiderivative separately, we obtain

$$\frac{3^x}{\ln 3} - \frac{1}{4}\cos(4x) - 2\ln|x| + c,$$

using the appropriate formulas.

(c) $\displaystyle\int \frac{t^3+t}{\sqrt{t^4+2t^2+7}}\, dt$

Answer: This is a u-substitution with $u = t^4+2t^2+7$. Then $du = 4t^3+4t dt = 4(t^3+t)dt$. Multiplying the integral by 4 on the inside and $1/4$ on the outside, the integral transforms to

$$\frac{1}{4}\int \frac{1}{\sqrt{u}}\, du \;=\; \frac{1}{4}\int u^{-1/2}\, du \;=\; \frac{1}{4}\cdot 2u^{1/2} + c \;=\; \frac{1}{2}u^{1/2} + c\,.$$

Converting back into the original variable gives

$$\frac{1}{2}\sqrt{t^4+2t^2+7} + c\,.$$

(d) $\displaystyle\int \frac{x^2}{x^6+1}\, dx$ **Hint:** Let $u = x^3$.

Answer: Letting $u = x^3$, we have $du = 3x^2\, dx$ and $x^6 = u^2$. Multiplying the integral by 3 on the inside and $1/3$ on the outside, the integral transforms to

$$\frac{1}{3}\int \frac{1}{u^2+1}\, du \;=\; \frac{1}{3}\tan^{-1}(u) + c\,.$$

Converting back into the original variable gives $\dfrac{1}{3}\tan^{-1}(x^3) + c\,.$

(e) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos(2\theta)\, e^{\sin(2\theta)}\, d\theta$

Answer: This is a u-substitution with $u = \sin(2\theta)$. Then $du = 2\cos(2\theta)\, d\theta$. Also, if $\theta = -\pi/4$, then $u = \sin(-\pi/2) = -1$ and if $\theta = \pi/4$, then $u = \sin(\pi/2) = 1$. Multiplying the integral by 2 on the inside and $1/2$ on the outside, the integral transforms to

$$\frac{1}{2}\int_{-1}^{1} e^u\, du \;=\; \frac{1}{2}e^u\Big|_{-1}^{1} \;=\; \frac{1}{2}\left(e - e^{-1}\right) \;=\; \frac{1}{2}\left(e - \frac{1}{e}\right)\,.$$

(f) $\displaystyle\int_0^1 \frac{(\tan^{-1} x)^3}{1+x^2}\, dx$

Answer: This is a u-substitution with $u = \tan^{-1} x$. Then $du = 1/(1+x^2)\, dx$. Also, if $x = 0$, then $u = \tan^{-1}(0) = 0$ and if $x = 1$, then $u = \tan^{-1}(1) = \pi/4$. Therefore, the integral transforms to

$$\int_0^{\pi/4} u^3\, du \;=\; \frac{1}{4}u^4\Big|_0^{\pi/4} \;=\; \frac{1}{4}\left(\frac{\pi^4}{4^4} - 0\right) \;=\; \frac{\pi^4}{1024}\,.$$

4. Suppose that the acceleration of a particle traveling along a line is given by

$$a(t) = e^{3t} - 4t.$$

If the initial velocity is $v(0) = 4$ and the initial position is $s(0) = 1$, find the position function $s(t)$.

Answer:

To find $v(t)$ we compute the antiderivative of the acceleration. Recall that

$$\int e^{kt}\, dt \;=\; \frac{1}{k}e^{kt} + c\,,$$

which is true for any constant k (check it with the chain rule.) Thus, we have that

$$v(t) \;=\; \frac{1}{3}e^{3t} - 2t^2 + c\,.$$

Since $v(0) = 4$, we find that $4 = 1/3 - 0 + c$, which implies that $c = 11/3$. Thus,

$$v(t) \;=\; \frac{1}{3}e^{3t} - 2t^2 + 11/3\,.$$

Next, we compute another antiderivative to find the position function $s(t)$. This gives

$$s(t) \;=\; \frac{1}{9}e^{3t} - \frac{2}{3}t^3 + \frac{11}{3}t + c\,.$$

Finally, using the initial position $s(0) = 1$, we have that $1 = 1/9 - 0 + 0 + c$, which implies that $c = 8/9$. The final answer is

$$s(t) \;=\; \frac{1}{9}e^{3t} - \frac{2}{3}t^3 + \frac{11}{3}t + \frac{8}{9}\,.$$

5. Evaluate $\displaystyle\int_0^{5/4} \frac{1}{\sqrt{25-4x^2}}\, dx$ using the substitution $u = \dfrac{2}{5}x$. Give the **exact** answer (no decimals).

Answer: Letting $u = \frac{2}{5}x$, we have $x = \frac{5}{2}u$ and $dx = \frac{5}{2}\, du$. Then,

$$\sqrt{25-4x^2} \;=\; \sqrt{25 - 4\left(\frac{5}{2}u\right)^2} \;=\; \sqrt{25 - 4\cdot\frac{25}{4}u^2}$$

$$=\; \sqrt{25-25u^2} \;=\; \sqrt{25(1-u^2)} \;=\; 5\sqrt{1-u^2}\,.$$

Also, if $x = 0$, then $u = 0$, and if $x = 5/4$, then $u = \frac{2}{5}\cdot\frac{5}{4} = 1/2$.

Applying the above calculations, the integral transforms to

$$\int_0^{1/2} \frac{1}{5}\cdot\frac{1}{\sqrt{1-u^2}}\cdot\frac{5}{2}\, du \;=\; \frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\, du$$

$$=\; \frac{1}{2}\sin^{-1} u\Big|_0^{1/2} \;=\; \frac{1}{2}\left(\sin^{-1}(1/2) - \sin^{-1}(0)\right) \;=\; \frac{\pi}{12}\,,$$

since $\sin^{-1}(1/2) = \pi/6$ and $\sin^{-1}(0) = 0$.

6. **Calculus Potpourri:**

(a) Suppose that $\int_{-3}^{0} f(x)\,dx = 5$ and $\int_{0}^{6} f(x)\,dx = 3$, and that $f(x)$ is an **odd** continuous function. Find the value of $\int_{3}^{6} 4f(x)\,dx$.

Answer: Since f is an odd function, it is symmetric with respect to the origin. This means the integral of f over an interval on one side of the y-axis is equivalent to minus the integral of f over the reflection of that interval onto the other side of the axis. Thus, we have $\int_{0}^{3} f(x)\,dx = -5$ because $\int_{-3}^{0} f(x)\,dx = 5$. Using linearity, we have

$$\int_{0}^{6} f(x)\,dx \;=\; \int_{0}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx\,,$$

which gives

$$3 \;=\; -5 + \int_{3}^{6} f(x)\,dx \quad \text{or} \quad \int_{3}^{6} f(x)\,dx \;=\; 8\,.$$

Then, since constants pull out of integrals, we have

$$\int_{3}^{6} 4f(x)\,dx \;=\; 4\cdot 8 \;=\; 32\,.$$

(b) Find the value of $\int_{-3}^{3} 4\sqrt{9-x^2}\,dx$ by interpreting the definite integral in terms of area.

Answer: First note that if $y = \sqrt{9-x^2}$, then $y^2 = 9 - x^2$ or $x^2 + y^2 = 9$. This is the equation of a circle centered at the origin of radius 3. It follows that the integral is equal to 4 times the area of a semi-circle of radius 3. We have

$$\int_{-3}^{3} 4\sqrt{9-x^2}\,dx \;=\; 4\cdot\frac{1}{2}\pi(3)^2 \;=\; 18\pi\,.$$

(c) A particle travels in a straight line with velocity $v(t) = 3t-3$ m/s. Find the total distance traveled by the particle over the time interval $[0,4]$.

Answer: To find the total distance traveled, we compute $\int_{0}^{4} |v(t)|\,dt = \int_{0}^{4} |3t-3|\,dt$. In order to evaluate this integral, we need to determine where $v(t)$ is positive and where it is negative. But $v(t)$ is just a line with slope 3 and t-intercept at $t = 1$ (solve $3t - 3 = 0$). It is negative for $0 \le t < 1$ and positive for $1 < t \le 4$. Therefore,

$$\begin{aligned}
\int_{0}^{4} |3t-3|\,dt \;&=\; \int_{0}^{1} 3 - 3t\,dt \;+\; \int_{1}^{4} 3t - 3\,dt \\
&=\; 3t - \frac{3}{2}t^2\Big|_{0}^{1} + \frac{3}{2}t^2 - 3t\Big|_{1}^{4} \\
&=\; \left(3 - \frac{3}{2}\right) - 0 + 24 - 12 - \left(\frac{3}{2} - 3\right) \\
&=\; \frac{3}{2} + 12 + \frac{3}{2} \\
&=\; 15\text{ m.}
\end{aligned}$$

Note that we can also evaluate the integral by interpreting it as the area under the graph of $|v(t)|$. This gives two triangles of area $3/2$ and $27/2$ for a total of $30/2 = 15$.

(d) Find and simplify $\dfrac{d}{dx}\left(\displaystyle\int_{\sqrt{x}}^{2020} \tan(t^2+1)\,dt\right)$.

Answer: This is a problem using FTC, part 2. First flip the limits of integration and then apply FTC, part 2 as well as the chain rule. The solution is

$$\begin{aligned}
\frac{d}{dx}\left(\int_{\sqrt{x}}^{2020} \tan(t^2+1)\,dt\right) &= -\frac{d}{dx}\left(\int_{2020}^{\sqrt{x}} \tan(t^2+1)\,dt\right) \\
&= -\tan((\sqrt{x}\,)^2+1)\cdot\frac{d}{dx}(\sqrt{x}\,) \\
&= -\tan(x+1)\cdot\frac{1}{2}x^{-1/2} \\
&= -\frac{\tan(x+1)}{2\sqrt{x}}\,.
\end{aligned}$$