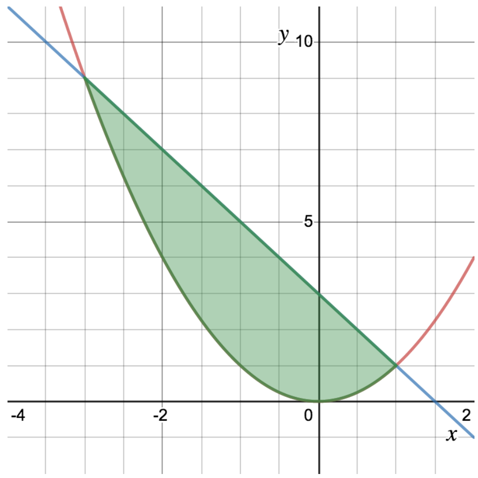
MATH 134 Calculus 2 with FUNdamentals

Exam #2 SOLUTIONS   April 3, 2020   Prof. Gareth Roberts

1. Let R be the region enclosed by the curves $y = x^2$ and $y = 3 - 2x$. (12 pts.)

(a) Sketch the region R in the xy-plane.

Answer:



The curves are a standard parabola and a line of slope -2 and y-intercept of 3. To find where the two curves intersect, we solve $x^2 = 3 - 2x$ or $x^2 + 2x - 3 = 0$. This equations factors as $(x+3)(x-1) = 0$, which means $x = -3$ or $x = 1$. Plugging back into either equation, we see that the line and parabola intersect at $(1, 1)$ and $(-3, 9)$.

(b) Find the area of the region R.

Answer: 32/3.

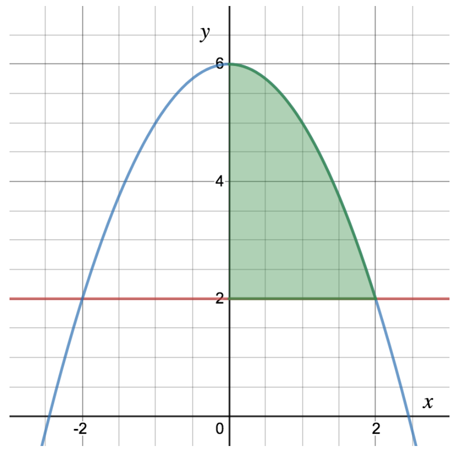
We find the area of R by integrating the difference of the top function (line) and the bottom one (parabola) from $x = -3$ to $x = 1$. We compute

$$\begin{aligned} A &= \int_{-3}^{1} 3 - 2x - x^2 \, dx \\ &= 3x - x^2 - \frac{x^3}{3} \Big|_{-3}^{1} \\ &= \left(3 - 1 - \frac{1}{3}\right) - (-9 - 9 + 9) \\ &= \frac{5}{3} + 9 \\ &= \frac{32}{3}. \end{aligned}$$

2. **Solids of Revolution:** Let A be the region in the first quadrant enclosed by the graphs of $x = 0$, $y = 2$, and $y = 6 - x^2$. (20 pts.)

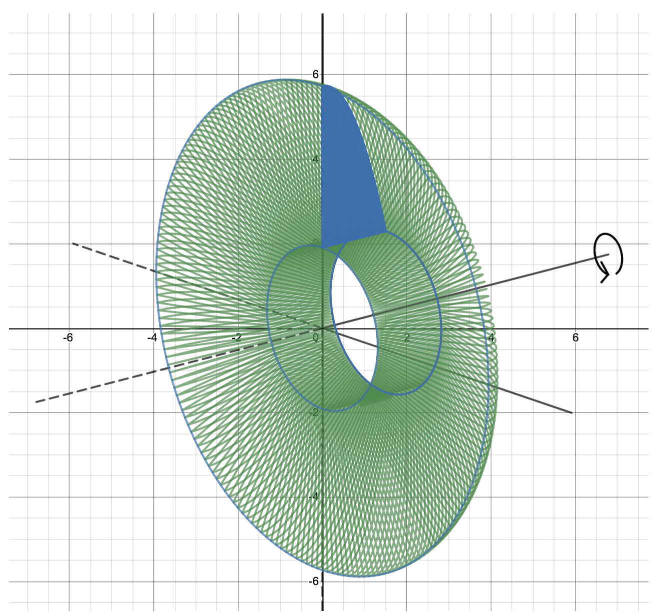
(a) Carefully sketch the region A in the xy-plane.

Answer:



The curves are a parabola opening downwards with vertex at $(0, 6)$ and a horizontal line. To find where the two curves intersect, we solve $6 - x^2 = 2$ or $x^2 = 4$. This means $x = 2$ since A is in the first quadrant. The curves intersect at $(2, 2)$.

(b) Find the volume of the solid of revolution obtained by rotating A about the x-axis. Give the **exact** answer (no decimals). Extra credit for drawing a good picture of the solid!



Answer: $192\pi/5$

We use the washer method since there is a gap between the region and the axis of rotation. The outer radius is $6 - x^2$ (distance between the parabola and the x-axis) and the inner radius is 2 (distance between the horizontal line and the x-axis). See the above figure, generated using `desmos.com` . Thus, the volume is given by

$$\begin{aligned}
\pi \int_0^2 (6-x^2)^2 - 2^2 \, dx &= \pi \int_0^2 36 - 12x^2 + x^4 - 4 \, dx \\
&= \pi \int_0^2 32 - 12x^2 + x^4 \, dx \\
&= \pi \left(32x - 4x^3 + \frac{x^5}{5} \Big|_0^2 \right) \\
&= \pi \left(64 - 32 + \frac{32}{5} \right) = \frac{192\pi}{5} .
\end{aligned}$$

(c) Find the volume of the solid of revolution obtained by rotating A about the y-axis. Give the **exact** answer (no decimals).

Answer: 8π
We use the disc method because there is no gap between the region and the axis of rotation. We integrate with respect to y because we are rotating about a vertical axis (circular cross sections are perpendicular to the y-axis). Since $y = 6 - x^2$, we have $x^2 = 6 - y$ or $x = \sqrt{6-y}$. This is the radius of each cross section as a function of y, where $2 \leq y \leq 6$. Thus, the volume is given by

$$\begin{aligned}
\int_2^6 \pi(\sqrt{6-y}\,)^2 \, dy &= \pi \int_2^6 6 - y \, dy \\
&= \pi \left(6y - \frac{y^2}{2} \Big|_2^6 \right) \\
&= \pi \left(36 - 18 - (12 - 2)\right) \\
&= \pi(18 - 10) \\
&= 8\pi .
\end{aligned}$$

3. Evaluate the following integrals using the appropriate method or combination of methods. (24 pts.)

(a) $\displaystyle\int \frac{4x+15}{x^2-5x} \, dx$

Answer: The denominator factors as $x(x-5)$ so this suggests partial fractions as an appropriate technique. We seek constants A and B such that

$$\frac{4x+15}{x(x-5)} = \frac{A}{x} + \frac{B}{x-5} .$$

Multiplying through by the LCD $x(x-5)$ gives

$$4x + 15 = A(x-5) + Bx.$$

Next we plug in the roots $x = 0$ and $x = 5$. Using $x = 0$ in the previous equation, we find $15 = -5A$ or $A = -3$. Likewise, setting $x = 5$ in the previous equation gives $35 = 5B$ or $B = 7$. Thus, the integral transforms into

$$\int \frac{-3}{x} + \frac{7}{x-5}\, dx \;=\; -3\ln|x| + 7\ln|x-5| + c\,.$$

(b) $\int t^6 \ln t\, dt$

Answer: Use integration by parts. Let $u = \ln t$ and $dv = t^6\, dt$. This leads to $du = \frac{1}{t}\, dt$ and $v = \frac{1}{7}t^7$. The integration by parts formula then yields

$$\begin{aligned}
\int t^6 \ln t\, dt &= \frac{1}{7}t^7 \ln t - \int \frac{1}{7}t^7 \cdot \frac{1}{t}\, dt \\
&= \frac{1}{7}t^7 \ln t - \frac{1}{7}\int t^6\, dt \\
&= \frac{1}{7}t^7 \ln t - \frac{1}{49}t^7 + c \\
&= \frac{t^7}{49}(7\ln t - 1) + c\,.
\end{aligned}$$

(c) $\int \cos^3 x \, \sin^2 x\, dx$

Answer: The first step is to factor out $\cos x$ and then use the identity $\cos^2 x = 1 - \sin^2 x$. We have

$$\begin{aligned}
\int \cos^3 x \, \sin^2 x\, dx &= \int \cos x \cdot \cos^2 x \cdot \sin^2 x\, dx \\
&= \int \cos x \cdot (1 - \sin^2 x) \sin^2 x\, dx \\
&= \int (1-u^2) \cdot u^2\, du \quad \text{using } u = \sin x, du = \cos x\, dx \\
&= \int u^2 - u^4 + c \\
&= \frac{u^3}{3} - \frac{u^5}{5} + c \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c\,.
\end{aligned}$$

4. Evaluate the integral $\int \frac{x^2}{x^2+9}\, dx$ using the trig substitution $x = 3\tan\theta$.

Hint: You will need to use the identity $\tan^2\theta + 1 = \sec^2\theta$ twice. (12 pts.)

Answer: Letting $x = 3\tan\theta$, we have $dx = 3\sec^2\theta\, d\theta$ and $x^2 = 9\tan^2\theta$. Also, using the trig identity $\tan^2\theta + 1 = \sec^2\theta$, the denominator simplifies as

$$x^2 + 9 \;=\; 9\tan^2\theta + 9 \;=\; 9(\tan^2\theta + 1) \;=\; 9\sec^2\theta\,.$$

We find

$$\begin{aligned}
\int \frac{x^2}{x^2+9}\,dx &= \int \frac{9\tan^2\theta}{9\sec^2\theta}\cdot 3\sec^2\theta\,d\theta \\
&= 3\int \tan^2\theta\,d\theta \\
&= 3\int \sec^2\theta - 1\,d\theta \qquad \text{(using } \tan^2\theta + 1 = \sec^2\theta \text{ again)} \\
&= 3(\tan\theta - \theta) + c \\
&= 3\tan\theta - 3\theta + c \\
&= x - 3\tan^{-1}\left(\frac{x}{3}\right) + c\,,
\end{aligned}$$

where the final step follows from $x = 3\tan\theta$ and $\tan\theta = x/3$.

5. Consider the two integrals below. One of these can be computed using a u-substitution while the other requires trig substitution. Determine which is which and evaluate **both** integrals. (16 pts.)

(a) $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$ **(b)** $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$

Answer: Integral **(a)** can be done using trig sub, while integral **(b)** can be evaluated using a u-sub.

For **(a)**, let $x = 4\sin\theta$. Then we have $dx = 4\cos\theta\,d\theta$ and $x^2 = 16\sin^2\theta$. Also, using the fundamental trig identity $\cos^2\theta + \sin^2\theta = 1$, the denominator simplifies to

$$\sqrt{16 - 16\sin^2\theta} \;=\; \sqrt{16(1-\sin^2\theta)} \;=\; \sqrt{16\cos^2\theta} \;=\; 4\cos\theta\,.$$

We find

$$\begin{aligned}
\int \frac{1}{\sqrt{16-x^2}}\,dx &= \int \frac{1}{4\cos\theta}\cdot 4\cos\theta\,d\theta \\
&= \int 1\,d\theta \\
&= \theta + c \\
&= \sin^{-1}\left(\frac{x}{4}\right) + c\,,
\end{aligned}$$

where the final step follows from $x = 4\sin\theta$ and thus $\sin\theta = x/4$.

Integral **(b)** can be done using the u-substitution $u = 16 - x^2$. Then $du = -2x\,dx$ or $x\,dx = -\frac{1}{2}\,du$. We compute

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{1}{\sqrt{u}} \cdot -\frac{1}{2}\,du \\ &= -\frac{1}{2}\int u^{-1/2}\,du \\ &= -\frac{1}{2}\cdot 2u^{1/2} + c \\ &= -\sqrt{u} + c \\ &= -\sqrt{16-x^2} + c\,.\end{aligned}$$

6. **Calculus Potpourri:** (16 pts.)

(a) Find the average value of $f(x) = \sin^2 x$ over the interval $[0, \pi]$.

Answer: 1/2. The average value of $f(x)$ over $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$, so we need to calculate $\dfrac{1}{\pi}\displaystyle\int_0^\pi \sin^2 x\,dx$. The average value is

$$\begin{aligned}\frac{1}{\pi}\int_0^\pi \sin^2 x\,dx &= \frac{1}{\pi}\int_0^\pi \frac{1}{2}\left(1-\cos(2x)\right)\,dx \\ &= \frac{1}{2\pi}\int_0^\pi 1-\cos(2x)\,dx \\ &= \frac{1}{2\pi}\left(x - \frac{1}{2}\sin(2x)\Big|_0^\pi\right) \\ &= \frac{1}{2\pi}\left(\pi - \frac{1}{2}\sin(2\pi) - 0 + \frac{1}{2}\sin(0)\right) \\ &= \frac{1}{2\pi}(\pi - 0) \\ &= \frac{1}{2}\,.\end{aligned}$$

(b) The population of Vivitown has a radial density function of $\rho(r) = 15e^{-2r}$, where r is the distance (in kilometers) from the city center and ρ is measured in thousands of people per square kilometer. Calculate the number of people living within 5 kilometers of the center of Vivitown (round to the nearest whole number).

Answer: 23,550 people live within 5 kilometers of the center of Vivitown.

The population is found by integrating the density function times $2\pi r$ over the interval $[0, 5]$. The integral can be computed using integration by parts with $u = r$ and $dv =$

$e^{-2r}\,dr$. Then $du = dr$ and $v = -\frac{1}{2}e^{-2r}$. We have

$$\begin{aligned}
\int_0^5 2\pi r \cdot 15e^{-2r}\,dr &= 30\pi \int_0^5 re^{-2r}\,dr \\
&= 30\pi \left(-\frac{1}{2}re^{-2r} - \int_0^5 -\frac{1}{2}e^{-2r}\,dr \right) \\
&= 30\pi \left(-\frac{1}{2}re^{-2r} - \frac{1}{4}e^{-2r} \Big|_0^5 \right) \\
&= 30\pi \left(-\frac{5}{2}e^{-10} - \frac{1}{4}e^{-10} - (0 - \frac{1}{4}) \right) \\
&= 30\pi \left(-\frac{11}{4}e^{-10} + \frac{1}{4} \right) \\
&= \frac{15\pi}{2}\left(1 - 11e^{-10}\right) \\
&\approx 23.5502 \text{ thousand people or } 23{,}550\,.
\end{aligned}$$