MATH 134 Calculus 2 with FUNdamentals

Applications to Economics: Consumer and Producer Surplus

In this worksheet we will explore an important application of calculus in economics called **consumer and producer surplus**. This material is not covered in our textbook.

Consumer Surplus

We begin with the **demand function** $p(x)$, which gives the relationship between the price p of a particular product and the amount x of the product that consumers are willing to purchase at that price. Demand functions are usually *decreasing* : the lower the price, the greater the demand for the product, or alternatively, the higher the price, the less likely consumers will purchase it (see the red curve in Figure 1).

Unlike with many other applications of calculus, in economics the independent variable is often on the vertical axis. Here we are interpreting the quantity x as depending on the price p, so we should really write $x = x(p)$. However, since supply and demand curves are usually one-to-one functions, they are invertible, so there is no harm in writing $p = p(x)$ for a demand curve.

Suppose that $\overline{p}$ is the current price and that $\overline{x}$ is the quantity purchased at that price. What happens if we raise the price? Some consumers are willing to pay more. By purchasing the product at the lower price $\overline{p}$, they are saving money. For example, if the cost of a muffin is \$2, but you were willing to pay \$3, then your consumer surplus is \$1. The money saved is the difference between the price you were willing to pay, $p(x)$, and the actual price $\overline{p}$. The **Consumer Surplus** measures the total amount of money saved by all consumers who are willing to pay a higher price than $\overline{p}$ for the product. Graphically, it is the area between $p(x)$ and $\overline{p}$ from $x = 0$ to $x = \overline{x}$ (see the red region in Figure 1).

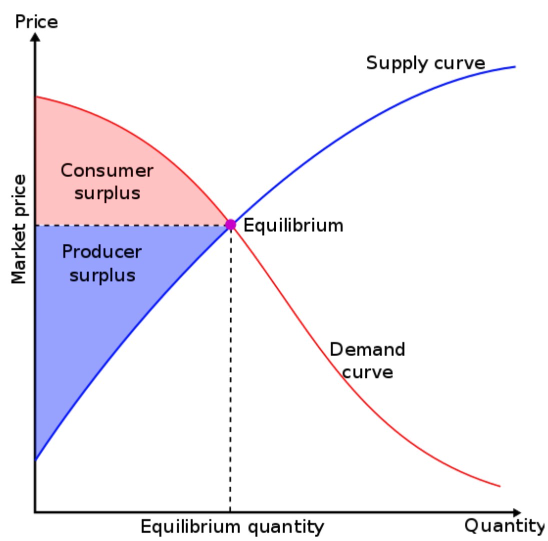

Figure 1: By User:SilverStar - Own work, CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=1450405

If $\overline{p}$ is the current price, and $\overline{x}$ is the number of units that will be purchased at that price, then the Consumer Surplus (CS) is found by computing the integral

$$\boxed{\text{CS} \;=\; \int_0^{\overline{x}} p(x) - \overline{p}\, dx\ .}$$

This is simply the formula for finding the area between two curves.

Example 1: Suppose that the demand function for producing a can of tennis balls is $p(x) = 20 - 0.05x$ and that the current price level is $\overline{p} = \$8$. Find the consumer surplus.

Answer: First, we need to find the value of $\overline{x}$ that corresponds to $\overline{p} = \$8$. Setting $p = 8$ and solving for x gives

$$8 = 20 - 0.05x \quad \Longrightarrow \quad 0.05x = 12 \quad \Longrightarrow \quad \overline{x} = 240.$$

Using the integral formula for consumer surplus, we find that

$$\text{CS} \;=\; \int_0^{240} (20 - 0.05x) - 8 \; dx \;=\; \int_0^{240} 12 - 0.05x \; dx \;=\; 12x - 0.025x^2 \Big|_0^{240} \;=\; \$1{,}440\,.$$

Producer Surplus

Now let's consider the problem from the producer's point of view. Let $s(x)$ denote the supply function, that is, for a given price s, x is the number of units of a product that manufacturers will produce at that price. Supply functions are typically increasing since the higher the price, the more likely companies want to produce it in order to make a greater profit (see the blue curve in Figure 1).

Once again, suppose that $\overline{p}$ is the current price and that $\overline{x}$ is the quantity purchased at that price. What happens if we lower the price? Some producers are still willing to supply the product. By selling the product at the higher price $\overline{p}$, they are making money. For example, if the cost of a muffin is \$2, but you were willing to sell it for \$1, then your producer surplus is \$1. The money earned is the difference between the price you were willing to pay, $s(x)$, and the actual price $\overline{p}$. The **Producer Surplus** measures the total amount of money gained by the producers who are willing to supply the product at a lower price than $\overline{p}$. Graphically, it is the area between $s(x)$ and $\overline{p}$ from $x = 0$ to $x = \overline{x}$ (see the blue region in Figure 1).

If $\overline{p}$ is the current price, and $\overline{x}$ is the number of units that will be purchased at that price, then the Producer Surplus (PS) is found by computing the integral

$$\boxed{\text{PS} \;=\; \int_0^{\overline{x}} \overline{p} - s(x) \; dx \;.}$$

Example 2: Suppose that the supply function for producing a can of tennis balls is $s(x) = 2 + 0.0002x^2$ and that the current price level is set at $\overline{p} = \$20$. Find the producer surplus.

Answer: First, we need to find the value of $\overline{x}$ that corresponds to $\overline{p} = \$20$. Setting $s = 20$ and solving for x gives

$$20 = 2 + 0.0002x^2 \quad \Longrightarrow \quad 0.0002x^2 = 18 \quad \Longrightarrow \quad x^2 = 90{,}000 \quad \Longrightarrow \quad \overline{x} = 300.$$

Using the integral formula for producer surplus, we find that

$$\text{PS} \;=\; \int_0^{300} 20 - (2 + 0.0002x^2) \; dx \;=\; \int_0^{300} 18 - 0.0002x^2 \; dx \;=\; 18x - \frac{0.0002}{3}x^3 \Big|_0^{300} \;=\; \$3{,}600\,.$$

Equilibrium Price

The total surplus is defined to be the sum of the consumer and producer surpluses. From the economic perspective, the goal of the system is to maximize the total surplus, thus making both consumers and producers happy. The price where this is achieved is called the **equilibrium value** and it is found where supply equals demand. In other words, the optimal price $\overline{p}$ is located at the intersection of the curves $p(x)$ and $s(x)$. The equilibrium quantity $\overline{x}$ is the value corresponding to $\overline{p}$ (see Figure 1).

Exercise 1: Suppose that the demand function for producing a can of tennis balls is $p(x) = 20 - 0.05x$ and the supply function is $s(x) = 2 + 0.0002x^2$.

(a) Find the equilibrium price $\overline{p}$ and quantity $\overline{x}$ by solving $p(x) = s(x)$.

(b) Find the value of the consumer surplus and producer surplus at the equilibrium price.

(c) Suppose that the price is set to $1 **greater** than the equilibrium price. Find the new value of the consumer and producer surplus and check that the total surplus (CS+PS) is less than the value obtained in part **(b)**. Use the same $\overline{x}$ in each integral (i.e., solve $p(x) = \overline{p} + 1$ first to obtain $\overline{x}$ and then use this as a limit of integration in both integrals).

Exercise 2: Repeat the instructions from Exercise 1 using $p(x) = \dfrac{15}{2x+1}$ and $s(x) = 4x + 3$. In this case, assume that x is measured in thousands of cans and the price p (and s) is measured in dollars per can.