MATH 134 Calculus 2 with FUNdamentals

Section 9.2: Newton's Law of Cooling

In this section we explore Newton's Law of Cooling, which is modeled by a simple linear differential equation.

Newton's Law of Cooling

The rate at which the temperature of an object cools (or warms) is proportional to the difference between its temperature and that of its surrounding medium:

$$\frac{dy}{dt} = k(y - A).$$

Here $y(t)$ is the temperature of the object at time t, A is the ambient temperature of the surrounding medium (a constant), and $k < 0$ is the **cooling constant**.

In most problems, the value of k is not given; it has to be determined from the given information. If the object has an initial temperature greater than A, then $dy/dt < 0$ and the object cools exponentially towards A. On the other hand, if the object is initially colder than A, then $dy/dt > 0$ and the object warms exponentially towards A.

Example 1: Suppose a gold ring at a temperature of 400°F is immersed in a tank of water which has a constant temperature of 60°F. After 4 minutes, the temperature of the ring is down to 100°F.

(a) What is the temperature of the ring after 6 minutes?

(b) What is the temperature of the ring after 15 minutes?

(c) What is the temperature of the ring after a very, very long time?

Answer: Let $y(t)$ be the temperature of the ring at time t in minutes, and let $A = 60$ be the ambient temperature. We are given two pieces of information about the temperature of the ring: $y(0) = 400$ (the initial temperature of the ring) and $y(4) = 100$ (the temperature after 4 minutes).

Using Newton's Law of Cooling, we have

$$\frac{dy}{dt} = k(y - 60)\,.$$

Using the Separation of Variables technique, we have

$$\begin{aligned} \frac{dy}{y-60} = k\,dt \quad &\Longrightarrow \quad \ln|y-60| = kt + c \\ &\Longrightarrow \quad |y-60| = e^{kt+c} = ce^{kt} \\ &\Longrightarrow \quad y - 60 = ce^{kt} \\ &\Longrightarrow \quad y = 60 + ce^{kt}\,. \end{aligned}$$

Now we find the values of c and k. Since $y(0) = 400$, we have $400 = 60 + ce^0 = 60 + c$. Therefore $c = 340$. Then $y(4) = 100$ implies $100 = 60 + 340e^{4k}$, which gives in turn

$$\frac{40}{340} = e^{4k} \quad \Longrightarrow \quad k = \frac{1}{4}\ln\left(\frac{2}{17}\right) \approx -0.5350165.$$

Thus we have found the formula for the temperature, $y(t) = 60 + 340e^{-0.5350165\,t}$.

(a) $y(6) = 60 + 340e^{-0.5350165 \cdot 6} \approx 73.72^\circ$ F.

(b) $y(15) = 60 + 340e^{-0.5350165 \cdot 15} \approx 60.11^\circ$ F.

(c) Since $\lim_{t \to \infty} y(t) = 60$, over the long term, the temperature is settling down to 60°F. This makes sense because this is the temperature of the water in the tank.

Exercises:

1. A cold metal bar at -20°C is submerged in a pool maintained at a temperature of 50°C. One minute later, the temperature of the bar is 10°C. How long will it take for the bar to reach a temperature of 30°C? What is the temperature of the bar after a very long time?

2. **Murder Mystery!** At 10:00 am you wander toward the pool table and find . . .

Miss Scarlet has been murdered with the candlestick in the billiard room! Stressed though you are by this terrible sight, you realize that you must catch the murderer. Fortunately, you have the presence of mind to measure the temperature of the body: 82.6°F. Then, you realize that you should round up the possible suspects:

- Colonel Mustard has an alibi from 3:00 pm – 5:00 pm and 9:30 pm – 5:00 am.
- Mr. Green has an alibi from 4:00 pm - 9:00 pm.
- Mrs. Peacock has an alibi from 8:00 pm – 1:00 am.

At 11:00 am you measure the temperature of the body again and find it is 81.7°F. The ambient temperature of the Billiard Room is kept at 72°F. Find the time of the murder to the nearest minute, assuming a healthy body temperature is 98.6°F. Which suspect should you detain for questioning?