MATH 134 Calculus 2 with FUNdamentals

Section 9.2: Newton's Law of Cooling

Solutions

Exercises:

1. A cold metal bar at $-20°$C is submerged in a pool maintained at a temperature of $50°$C. One minute later, the temperature of the bar is $10°$C. How long will it take for the bar to reach a temperature of $30°$C? What is the temperature of the bar after a very long time?

   Answer: Let $y(t)$ be the temperature of the bar (in celsius) at time t in minutes, and let $A = 50$ be the ambient temperature. We are given two pieces of information about the temperature of the bar: $y(0) = -20$ (the initial temperature of the bar) and $y(1) = 10$ (the temperature after one minute).

   Using Newton's Law of Cooling, we have

   $$\frac{dy}{dt} = k(y - 50)\,.$$

   Using the Separation of Variables technique, we have

   $$\begin{aligned} \frac{dy}{y-50} = k\,dt \implies \ln|y-50| &= kt + c \\ \implies |y-50| &= e^{kt+c} = ce^{kt} \\ \implies y - 50 &= ce^{kt} \\ \implies y &= 50 + ce^{kt}\,. \end{aligned}$$

   Now we find the values of c and k. Since $y(0) = -20$, we have $-20 = 50 + ce^0 = 50 + c$. Therefore $c = -70$. Then $y(1) = 10$ implies $10 = 50 - 70e^k$, which gives in turn

   $$\frac{40}{70} = e^k \implies k = \ln\left(\frac{4}{7}\right) \approx -0.5596158.$$

   Thus we have found the formula for the temperature, $y(t) = 50 - 70e^{-0.5596158\,t}$.

   To find when the temperature reaches $30°$C, we set $y = 30$ and solve for t. We have

   $$\begin{aligned} 30 = 50 - 70e^{-0.5596158\,t} &\implies \frac{20}{70} = e^{-0.5596158\,t} \\ &\implies \ln\left(\frac{2}{7}\right) = -0.5596158\,t \\ &\implies t = -\frac{1}{0.5596158}\ln\left(\frac{2}{7}\right) \approx 2.24 \text{ minutes} \end{aligned}$$

   Since $\lim\limits_{t\to\infty} y(t) = 50$, over the long term, the temperature of the bar is settling down to $50°$F. This makes sense because this is the temperature of the pool.

2. **Murder Mystery!** At 10:00 am you wander toward the pool table and find that Miss Scarlet has been murdered with the candlestick in the billiard room! Stressed though you are by this terrible sight, you realize that you must catch the murderer. Fortunately, you have the presence of mind to measure the temperature of the body: 82.6°F. Then, you realize that you should round up the possible suspects:

- Colonel Mustard has an alibi from 3:00 pm – 5:00 pm and 9:30 pm – 5:00 am.
- Mr. Green has an alibi from 4:00 pm - 9:00 pm.
- Mrs. Peacock has an alibi from 8:00 pm – 1:00 am.

At 11:00 am you measure the temperature of the body again and find it is 81.7°F. The ambient temperature of the billiard room is kept at 72°F. Find the time of the murder to the nearest minute, assuming a healthy body temperature is 98.6°F. Which suspect should you detain for questioning?

Answer: The key to this problem is setting up the initial conditions correctly. Let $y(t)$ be the temperature of the body (in degrees fahrenheit) at time t in hours, and let $A = 72$ be the ambient temperature of the room. We will define $t = 0$ to correspond to the time 10:00 am. Then $t = 1$ means 11:00 am. Since the goal is to find the time of death, we will work backwards and look for a *negative* time t at which $y = 98.6$. Our initial conditions are:

- $y(0) = 82.6$ (temperature of the body at 10:00 am)
- $y(1) = 81.7$ (temperature of the body at 11:00 am)
- **Goal:** Solve $y(t) = 98.6$ for t ($t < 0$ means subtract this time from 10:00 am)

Using Newton's Law of Cooling, we have

$$\frac{dy}{dt} = k(y - 72)\,.$$

Using the Separation of Variables technique, we have

$$\begin{aligned} \frac{dy}{y-72} = k\,dt \implies \ln|y-72| &= kt + c \\ \implies |y-72| &= e^{kt+c} = ce^{kt} \\ \implies y - 72 &= ce^{kt} \\ \implies y &= 72 + ce^{kt}\,. \end{aligned}$$

Now we find the values of c and k. Since $y(0) = 82.6$, we have $82.6 = 72 + ce^0 = 72 + c$. Therefore $c = 10.6$. Then $y(1) = 81.7$ implies $81.7 = 72 + 10.6e^k$, which gives in turn

$$\frac{9.7}{10.6} = e^k \quad \implies \quad k = \ln\left(\frac{9.7}{10.6}\right) \approx -0.08872812.$$

Thus we have found the formula for the temperature, $y(t) = 72 + 10.6e^{-0.08872812\,t}$.

To find when the temperature of the body *was* 98.6°C, we set $y = 98.6$ and solve for t. We have

$$\begin{aligned} 98.6 &= 72 + 10.6e^{-0.08872812\,t} \implies \frac{26.6}{10.6} = e^{-0.08872812\,t} \\ &\implies \ln\left(\frac{26.6}{10.6}\right) = -0.08872812\,t \\ &\implies t = -\frac{1}{0.08872812}\ln\left(\frac{26.6}{10.6}\right) \approx -10.3694 \text{ hours} \end{aligned}$$

This means the murder occurred 10.3694 hours *before* 10:00 am. Since $0.3694 \cdot 60 \approx 22$, this is 10 hours and 22 minutes before 10:00 am or 11:38 pm. It looks like Mr. Green is the murderer!