MATH 134 Calculus 2 with FUNdamentals

Section 8.1: Arc Length

Solutions

Exercises: For #1–2, compute the arc length of the graph of each function over the given interval.

1. $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

   Answer: $\dfrac{8}{27}\left(10\sqrt{10} - 1\right)$

   We have $f'(x) = \frac{3}{2}x^{1/2}$, so that $1 + (f'(x))^2 = 1 + \frac{9}{4}x$. The arc length integral can be computed with a u-substitution with $u = 1 + \frac{9}{4}x$. We have

$$\begin{aligned} L &= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \int_1^{10} u^{1/2} \cdot \frac{4}{9}\, du \qquad (u = 1 + \tfrac{9}{4}x,\ du = \tfrac{9}{4}dx,\ dx = \tfrac{4}{9}du) \\ &= \frac{4}{9}\int_1^{10} u^{1/2}\, du \\ &= \frac{4}{9} \cdot \frac{2}{3} u^{3/2}\Big|_1^{10} \\ &= \frac{8}{27}\left(10\sqrt{10} - 1\right) \end{aligned}$$

2. $f(x) = x^3 + \frac{1}{12}x^{-1}$ from $x = 1$ to $x = 2$.

   Answer: $\dfrac{169}{24}$.

   The key to this problem is to write $1 + [f'(x)]^2$ as a perfect square. We compute $f'(x) = 3x^2 - \frac{1}{12}x^{-2}$. Then,

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + 9x^4 + \frac{1}{144}x^{-4} - \frac{1}{2} \\ &= 9x^4 + \frac{1}{2} + \frac{1}{144}x^{-4} \\ &= \left(3x^2 + \frac{1}{12}x^{-2}\right)^2. \end{aligned}$$

This implies that the arc length is

$$\begin{aligned}
\int_1^2 \sqrt{\left(3x^2 + \frac{1}{12}x^{-2}\right)^2}\, dx &= \int_1^2 3x^2 + \frac{1}{12}x^{-2}\, dx \\
&= x^3 - \frac{1}{12}x^{-1}\Big|_1^2 \\
&= 8 - \frac{1}{24} - \left(1 - \frac{1}{12}\right) \\
&= 7 + \frac{1}{24} \\
&= \frac{169}{24}\,.
\end{aligned}$$

3. Verify that the circumference of the unit circle is 2π by computing the arc length of the curve $y = \sqrt{1-x^2}$ from $x = -1$ to $x = 1$.

Answer: First write the function as $y = (1-x^2)^{1/2}$. Using the chain rule, we have $\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2} \cdot -2x = -x(1-x^2)^{-1/2}$. Then,

$$\begin{aligned}
1 + [f'(x)]^2 &= 1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2 \\
&= 1 + \frac{x^2}{1-x^2} \\
&= \frac{1}{1-x^2}\,.
\end{aligned}$$

This implies that the arc length is

$$\begin{aligned}
\int_{-1}^1 \sqrt{\frac{1}{1-x^2}}\, dx &= \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \sin^{-1}(x)\Big|_{-1}^1 \\
&= \sin^{-1}(1) - \sin^{-1}(-1) \\
&= \frac{\pi}{2} - (-\frac{\pi}{2}) \\
&= \pi\,.
\end{aligned}$$

This shows that the length of the top half of the unit circle is π, which implies that the full unit circle has a length (i.e., circumference) of 2π.

Note that the integral above is technically improper because the integrand is undefined at -1 and 1. However, using the standard limit definition (replacing the "bad" points with b) will lead to the same result since $\sin^{-1}(x)$ is left- and right-continuous at $x = 1$ and $x = -1$, respectively.