

MATH 134 Calculus 2 with FUNdamentals

Section 7.2: Trigonometric Integrals

This section focuses on integrals involving powers of $\cos x$ and $\sin x$. For these types of integrals, the key is to use the correct trig identity to simplify the integrand into a form where a u -sub or formula may be applied to compute the integral.

1) Odd Powers of $\cos x$ or $\sin x$

Example 1: Compute $\int \cos^5 x \, dx$ using the identity $\cos^2 x = 1 - \sin^2 x$.

Answer: When either $\cos x$ or $\sin x$ is raised to an **odd** power, break off one of the $\cos x$ or $\sin x$ terms and then use the identity above to replace the even power. For this example, we write

$$\cos^5 x = \cos x \cdot \cos^4 x = \cos x \cdot (\cos^2 x)^2 = \cos x(1 - \sin^2 x)^2 = (1 - 2\sin^2 x + \sin^4 x) \cos x.$$

Now we can use a u -substitution with $u = \sin x$ and $du = \cos x \, dx$. We obtain

$$\int \cos^5 x \, dx = \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx = \int 1 - 2u^2 + u^4 \, du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c.$$

Returning to the original variable, the solution is $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$.

A similar strategy will work when $\sin x$ is raised to an odd power. In this case, use the identity $\sin^2 x = 1 - \cos^2 x$ and let $u = \cos x$.

2) Even Powers of $\cos x$ and $\sin x$

When **both** $\cos x$ and $\sin x$ are raised to an even power, we can use one of the double angle formulas

$$\boxed{\cos^2 x = \frac{1}{2}(1 + \cos(2x)) \quad \text{or} \quad \sin^2 x = \frac{1}{2}(1 - \cos(2x))}. \quad (1)$$

These identities can be derived from the double angle formulas for cosine:

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

Note that the last two expressions can be obtained from the first by using $\cos^2 x + \sin^2 x = 1$.

Example 2: Evaluate $\int_0^{\pi/4} \cos^2 \theta \, d\theta$.

Answer: We use the first equation in (1).

$$\int_0^{\pi/4} \cos^2 \theta \, d\theta = \int_0^{\pi/4} \frac{1}{2}(1 + \cos(2\theta)) \, d\theta = \frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta)) \Big|_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi}{8} + \frac{1}{4} = \frac{2 + \pi}{8}.$$

Exercises

1. Compute $\int \sin^3 x \, dx$ using the identity $\sin^2 x = 1 - \cos^2 x$.

2. Evaluate $\int \sin^4 x \cdot \cos^3 x \, dx$.

3. Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 \theta \, d\theta$.

4. Evaluate $\int \cos^4 x \, dx$.

Hint: Write $\cos^4 x = (\cos^2 x)^2$ and use the first equation in (1).

5. Compute $\int_0^{\pi/2} \sin^5 t \, dt$.