# MATH 134 Calculus 2 with FUNdamentals

## Section 7.5: Partial Fractions

This section focuses on integrals involving rational functions $p(x)/q(x)$ where $p$ and $q$ are polynomials, and $p$ has a lower degree than $q$. The main idea is to break the fraction up into pieces that are easily integrated. Splitting the fraction into smaller pieces is called a **partial fraction decomposition**. We will restrict our attention to denominators that have distinct linear factors.

**Example 1:** Use the method of partial fractions to evaluate $\displaystyle\int \frac{7x+13}{x^2+5x-14}\,dx$.

**Answer:** Notice that the denominator factors as $(x+7)(x-2)$. To compute the partial fraction decomposition of the integrand, we seek constants $A$ and $B$ such that

$$\frac{7x+13}{x^2+5x-14} \;=\; \frac{A}{x+7} \;+\; \frac{B}{x-2}\,. \tag{1}$$

It turns out that whenever the denominator has distinct linear factors, it is always possible to find these special constants $A$ and $B$ (they are unique). We begin by multiplying both sides of equation (1) by the least common denominator $(x+7)(x-2)$:

$$(x+7)(x-2)\left(\frac{7x+13}{x^2+5x-14}\right) \;=\; (x+7)(x-2)\left(\frac{A}{x+7}+\frac{B}{x-2}\right).$$

After cancelling, this gives

$$7x+13 \;=\; A(x-2)+B(x+7). \tag{2}$$

This equation needs to be satisfied **for any** $x$. Here is a useful trick: To find $A$ and $B$, **plug in the roots** $x=2$ and $x=-7$. Plugging in $x=2$ into equation (2) gives $27 = A\cdot 0 + B\cdot 9$, which implies $B=3$. Plugging in $x=-7$ into equation (2) gives $-36 = A\cdot(-9)+B\cdot 0$, which implies $A=4$. To compute the integral, we break the fraction into two pieces, each of which can be integrated using simple $u$-substitutions:

$$\int \frac{7x+13}{x^2+5x-14}\,dx \;=\; \int \frac{4}{x+7}+\frac{3}{x-2}\,dx \;=\; 4\ln|x+7|+3\ln|x-2|+c\,.$$

In the final step, the two integrals are computed with $u$-substitutions $u=x+7$ and $u=x-2$, respectively. Note that $du=dx$ in both cases, so each integral is of the form $k\displaystyle\int \frac{1}{u}\,du = k\ln|u|$.

The technique above generalizes to other settings, provided we know how to compute the partial fraction decomposition of the integrand. Below are the relevant partial fraction decompositions we will make use of. The goal is to find the values of the unknown constants $A, B$, and $C$. After multiplying through by the LCD, plugging in the roots for $x$ is always a good idea.

1. $\displaystyle\frac{p(x)}{(x-a)(x-b)} \;=\; \frac{A}{x-a}+\frac{B}{x-b}$ Two Distinct Linear Factors

2. $\displaystyle\frac{p(x)}{(x-a)(x-b)(x-c)} \;=\; \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$ Three Distinct Linear Factors

**Exercises:** Evaluate each of the following integrals using the method of partial fractions.

1. $\displaystyle\int \frac{3x+20}{x^2+4x}\,dx$

2. $\displaystyle\int \frac{3x-25}{x^2+2x-15}\,dx$

3. $\displaystyle\int \frac{46-x^2}{(x+1)(x-2)(x-4)}\,dx$

4. $\displaystyle\int \frac{2x^2+18}{x^3-x}\,dx$