MATH 134 Calculus 2 with FUNdamentals

Section 7.5: Partial Fractions   SOLUTIONS

Exercises: Evaluate each of the following integrals using the method of partial fractions.

1. $\displaystyle\int \frac{3x+20}{x^2+4x}\,dx$

   Answer: Notice that the denominator factors as $x^2+4x = x(x+4)$. To compute the partial fraction decomposition, we seek constants A and B such that

$$\frac{3x+20}{x^2+4x} \;=\; \frac{A}{x} \;+\; \frac{B}{x+4}\,.$$

   Multiply both sides by the least common denominator $x(x+4)$:

$$x(x+4)\left(\frac{3x+20}{x^2+4x}\right) \;=\; x(x+4)\left(\frac{A}{x}+\frac{B}{x+4}\right).$$

   After cancelling, this gives

$$3x+20 \;=\; A(x+4)+Bx.$$

   To find A and B, plug in the roots of the original denominator: $x=0$ and $x=-4$. Plugging in $x=0$ gives $20 = A\cdot 4 + B\cdot 0$, which implies $A=5$. Plugging in $x=-4$ gives $8 = A\cdot 0 + B\cdot(-4)$, which implies $B=-2$.

   To compute the integral, we break the fraction into two pieces:

$$\int \frac{3x+20}{x^2+4x}\,dx = \int \frac{5}{x} + \frac{-2}{x+4}\,dx = 5\ln|x| - 2\ln|x+4| + c\,.$$

   In the final step, the second integral is computed with the u-substitution $u = x+4$. Note that $du = dx$ so the integral becomes $-2\displaystyle\int \frac{1}{u}\,du = -2\ln|u| + c$.

2. $\displaystyle\int \frac{3x-25}{x^2+2x-15}\,dx$

   Answer: Notice that the denominator factors as $x^2+2x-15 = (x+5)(x-3)$. To compute the partial fraction decomposition, we seek constants A and B such that

$$\frac{3x-25}{x^2+2x-15} \;=\; \frac{A}{x+5} \;+\; \frac{B}{x-3}\,.$$

   Multiply both sides by the least common denominator $(x+5)(x-3)$:

$$(x+5)(x-3)\left(\frac{3x-25}{x^2+2x-15}\right) \;=\; (x+5)(x-3)\left(\frac{A}{x+5}+\frac{B}{x-3}\right).$$

   After cancelling, this gives

$$3x-25 \;=\; A(x-3)+B(x+5).$$

To find A and B, plug in the roots of the original denominator: $x = -5$ and $x = 3$. Plugging in $x = -5$ gives $-40 = A \cdot (-8) + B \cdot 0$, which implies $A = 5$. Plugging in $x = 3$ gives $-16 = A \cdot 0 + B \cdot 8$, which implies $B = -2$.

To compute the integral, we break the fraction into two pieces, each of which can be integrated using simple u-substitutions:

$$\int \frac{3x - 25}{x^2 + 2x - 15}\,dx = \int \frac{5}{x+5} + \frac{-2}{x-3}\,dx = 5\ln|x+5| - 2\ln|x-3| + c\,.$$

In the final step, the integrals are computed with the u-substitutions $u = x+5$ and $u = x-3$, respectively.

3. $\displaystyle\int \frac{46 - x^2}{(x+1)(x-2)(x-4)}\,dx$

Answer: In this case the denominator is already factored for us; however, notice that there are three roots. In this case, we seek constants A, B, and C such that

$$\frac{46 - x^2}{(x+1)(x-2)(x-4)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-4}\,.$$

Multiply both sides by the least common denominator $(x+1)(x-2)(x-4)$:

$$(x+1)(x-2)(x-4)\left(\frac{46 - x^2}{(x+1)(x-2)(x-4)}\right) = (x+1)(x-2)(x-4)\left(\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-4}\right).$$

After cancelling, this gives

$$46 - x^2 = A(x-2)(x-4) + B(x+1)(x-4) + C(x+1)(x-2).$$

To find $A, B,$ and C plug in the roots of the original denominator: $x = -1$, $x = 2$, and $x = 4$. Plugging in $x = -1$ gives $45 = A \cdot 15 + B \cdot 0 + C \cdot 0$, which implies $A = 3$. Plugging in $x = 2$ gives $42 = A \cdot 0 + B \cdot (-6) + C \cdot 0$, which shows $B = -7$. Plugging in $x = 4$ gives $30 = A \cdot 0 + B \cdot 0 + C \cdot 10$, which yields $C = 3$.

To compute the integral, we break the fraction into three pieces, each of which can be integrated using simple u-substitutions:

$$\begin{aligned}
\int \frac{46 - x^2}{(x+1)(x-2)(x-4)}\,dx &= \int \frac{3}{x+1} + \frac{-7}{x-2} + \frac{3}{x-4}\,dx \\
&= 3\ln|x+1| - 7\ln|x-2| + 3\ln|x-4| + c \\
&= 3\ln|(x+1)(x-4)| - 7\ln|x-2| + c\,,
\end{aligned}$$

where the final step uses the property $\ln(a) + \ln(b) = \ln(ab)$.

4. $\displaystyle\int \frac{2x^2 + 18}{x^3 - x}\,dx$

Answer: First, factor the denominator as $x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$. We seek constants A, B, and C such that

$$\frac{2x^2 + 18}{x^3 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\,.$$

Multiply both sides by the least common denominator $x(x-1)(x+1)$:

$$x(x-1)(x+1)\left(\frac{2x^2+18}{x^3-x}\right) \;=\; x(x-1)(x+1)\left(\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}\right).$$

After cancelling, this gives

$$2x^2+18 \;=\; A(x-1)(x+1)+Bx(x+1)+Cx(x-1).$$

To find A, B, and C plug in the roots of the original denominator: $x=0$, $x=1$, and $x=-1$. Plugging in $x=0$ gives $18 = A\cdot(-1)+B\cdot 0+C\cdot 0$, which implies $A=-18$. Plugging in $x=1$ gives $20 = A\cdot 0+B\cdot 2+C\cdot 0$, which shows $B=10$. Plugging in $x=-1$ gives $20 = A\cdot 0+B\cdot 0+C\cdot 2$, which yields $C=10$.

To compute the integral, we break the fraction into three pieces, each of which can be integrated using simple u-substitutions:

$$\begin{aligned}\int \frac{2x^2+18}{x^3-x}\,dx \;&=\; \int \frac{-18}{x}+\frac{10}{x-1}+\frac{10}{x+1}\,dx \\ &=\; -18\ln|x|+10\ln|x-1|+10\ln|x+1|+c \\ &=\; 10\ln|x^2-1|-18\ln|x|+c\,,\end{aligned}$$

where the final step uses the property $\ln(a)+\ln(b)=\ln(ab)$.