MATH 134 Calculus 2 with FUNdamentals

Section 7.6: Strategies for Integration

This section summarizes all the different strategies for integration we have learned thus far. When given a random integral, it is important to know which approach is likely to be the most successful. There are often several techniques that will work, but finding the best approach saves time and effort. Here is a list of techniques and a strategy for integration:

1. Determine whether the integral is possible to compute directly using a known formula (e.g., power rule, exponential, trig function, inverse trig., etc.).

2. Simplify the integrand if possible. Sometimes the power rule can be applied after simplification.

3. Look for a **u-substitution.** Is there a function u and it's derivative du present in the integrand?

4. Try **integration by parts** if there is a product of two functions, one of which is relatively simple (e.g., x or x^2). Typically we take u to be a power, logarithmic, or inverse trig function. Remember to take the *antiderivative* when going from dv to v.

5. If the integrand is of the form $\cos^m x \sin^n x$ where one of m or n is odd, pull out one trig factor from the odd power (e.g., $\cos x$) and make a u-substitution with u equal to the *other* trig function (e.g., $u = \sin x$). Use the identity $\cos^2 x + \sin^2 x = 1$ to write the integral in terms of your u.

6. If the integrand contains only even powers of $\cos x$ or $\sin x$, use **trig identities** such as $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ or $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$ to simplify the integral into simple sine and cosine functions.

7. If the integral contains $\sqrt{a^2 - x^2}$, use the **trig substitution** $x = a \sin\theta$. If the integral contains $\sqrt{x^2 + a^2}$, or $x^2 + a^2$ in the denominator, use $x = a \tan\theta$ and the identity $1 + \tan^2\theta = \sec^2\theta$. The identity $\sin(2\theta) = 2\sin\theta\cos\theta$ is often useful.

8. If the integral is a rational function of the form $p(x)/q(x)$, try factoring the denominator and using the method of **partial fractions.**

9. Sometimes two methods need to be combined. For example, a u-substitution may be necessary before integration by parts can be applied, or a trig substitution may lead to an integral that can be computed using a u-sub. **Note:** In general, most integrals do *not* have antiderivatives that can be written in closed form (a simple expression).

Example 1: Trig sub versus u-sub

Consider the integral $\displaystyle\int \frac{x}{x^2+4}\, dx$. Seeing the x^2+4 in the denominator, we try the trig substitution $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$ and $x^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$. The integral transforms to

$$\int \frac{2\tan\theta}{4\sec^2\theta} \cdot 2\sec^2\theta\, d\theta \;=\; \int \tan\theta\, d\theta \;=\; \int \frac{\sin\theta}{\cos\theta}\, d\theta\,.$$

The integral can now be computed using the u-substitution $u = \cos\theta, du = -\sin\theta\, d\theta$. This gives

$$-\int \frac{1}{u}\, du \;=\; -\ln|u| + c \;=\; -\ln|\cos\theta| + c \;=\; -\ln\left|\frac{2}{\sqrt{x^2+4}}\right| + c\,,$$

where we use right-triangle trig and SOH-CAH-TOA to find $\cos\theta$ from $\tan\theta = x/2$.

On the other hand, we could also compute the original integral by doing a u-sub with $u = x^2 + 4$. Then $du = 2x\,dx$ is sitting in the numerator of the fraction (except for a factor of 2). We have

$$\int \frac{x}{x^2+4}\,dx \;=\; \int \frac{1}{u}\cdot\frac{du}{2} \;=\; \frac{1}{2}\int \frac{1}{u}\,du \;=\; \frac{1}{2}\ln|u| + c \;=\; \frac{1}{2}\ln|x^2+4| + c\,.$$

Notice how much easier it is to evaluate the integral with a u-substitution. Using the properties $\ln(\frac{a}{b}) = \ln a - \ln b$ and $\ln a^b = b\ln a$, you can check that our first solution is equivalent to the second (the $-\ln 2$ can be incorporated into the constant c).

Exercises: Evaluate each of the following integrals by choosing an appropriate technique(s) of integration.

1. $\displaystyle\int x\sqrt{x^2+1}\,dx$

2. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$ **Hint:** Use the u-sub $u = x^2 + 1$ and write $x^3 = x^2 \cdot x$.

3. $\displaystyle\int t^2 e^{3t}\,dt$

4. $\displaystyle\int_0^{\pi/4} \sin^4 \theta \, d\theta$

5. $\displaystyle\int \frac{x^2}{(9-x^2)^{3/2}} \, dx$ **Hint:** After making a trig substitution, use the identity $\tan^2 \theta + 1 = \sec^2 \theta$ to evaluate the integral.

6. $\displaystyle\int \frac{24}{x^2-9} \, dx$