MATH 134 Calculus 2 with FUNdamentals

Section 6.3: Volumes of Revolution

In this section we learn how to find the volume of a 3D solid obtained by rotating a region (area) of the plane about a horizontal or vertical axis. Our goal is to find the volume of the resulting **solid of revolution**. There are two techniques we will learn depending on the type of cross section: the disc method and the washer method.

Example 1: The region bounded by the lines $y = 0, y = x$, and $x = 3$ is rotated 360° about the x-axis. Find the volume of the resulting solid of revolution.

Answer: First we draw the region in the xy-plane bounded by the three lines. The result is a right triangle with base and height each equal to 3. When this triangle is rotated about the x-axis, the resulting solid is a cone (see Figure 1).

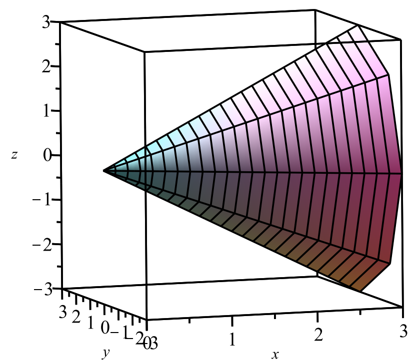


Figure 1: The solid of revolution (in this case a cone) obtained by rotating the area under $y = x$ from $x = 0$ to $x = 3$ about the x-axis.

To find the volume of the cone, we take cross sections perpendicular to the x-axis of width Δx. The cross sections are circles with radii given by the height of the function $f(x) = x$. In other words, the area of each cross section is $A(x) = \pi r^2 = \pi x^2$. If x is close to 0, then the area is small. As x increases from 0 to 3, the area of each cross section increases. Using the formula from Section 6.2, we compute the volume of the cone to be

$$V = \int_a^b A(x)\, dx = \int_0^3 \pi x^2\, dx = \pi \frac{x^3}{3}\bigg|_0^3 = 9\pi - 0 = 9\pi\,.$$

Note that this agrees with the result from the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$. The radius and height of the cone are each 3, so we obtain $V = \frac{1}{3}\pi \cdot 3^2 \cdot 3 = 9\pi$, as expected.

The Disc Method

The previous example suggest a technique for finding the volume of a solid of revolution when the cross sections are all discs with centers on the x-axis. Since the height of the function corresponds to the radius of each disc, we have the following formula (see Figure 2).

Disc method: Rotating the area under the graph of $f(x)$ between $x = a$ and $x = b$ about the x-axis (360° rotation) yields a solid of revolution with volume

$$\boxed{V = \int_a^b \pi (f(x))^2\, dx\,.} \tag{1}$$

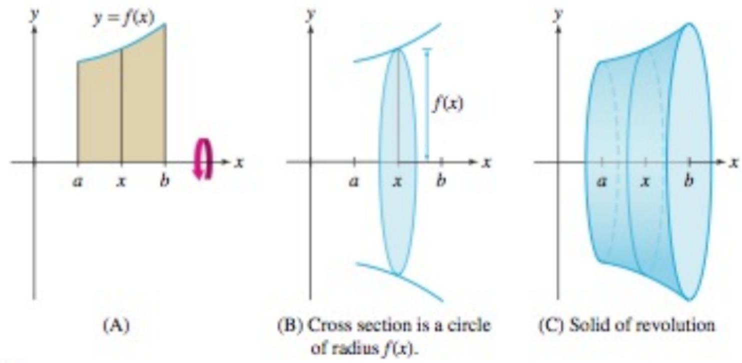
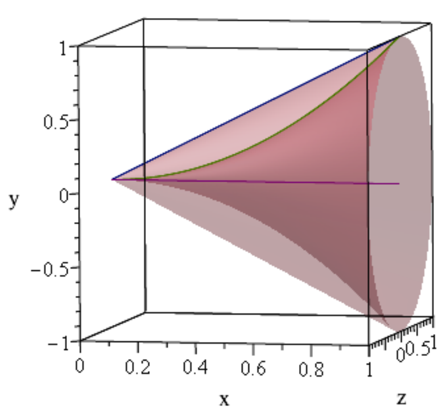


Figure 2: The disc method for finding the volume of a solid of revolution.

Exercise 1: Find the volume of the solid of revolution obtained by rotating about the x-axis the region under the curve $f(x) = e^{-x}$ between $x = 0$ and $x = 1$. Sketch the solid (use dashed lines for the parts of cross-sections that are not visible).

Example 2: The region bounded by the curves $y = x^2$ and $y = x$ is rotated about the x-axis. Find the volume of the resulting solid of revolution.

Answer:



The region and corresponding solid of revolution are shown on the left. Note that $y = x^2$ and $y = x$ intersect at $x = 0$ and $x = 1$. The solid can be viewed as removing the inner horn-shaped region (dark red) from the outer cone (light red). In this case the cross sections are annuli or **washers** perpendicular to the x-axis. The outer radius of the washer is $r = x$, while the inner radius is $r = x^2$. The area is found by computing the difference of the areas between the outer circle, πx^2, and the inner circle, $\pi(x^2)^2$.

We find the volume of the solid by summing up these areas times Δx:

$$V = \int_0^1 \pi(x)^2 - \pi(x^2)^2 \, dx = \pi \int_0^1 x^2 - x^4 \, dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \bigg|_0^1 \right) = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$$

The Washer Method

As with the previous example, if the area being rotated is *not* adjacent to the axis of rotation, then the cross sections will always be washers. In this case, we subtract the area of the inner circle from the outer circle and integrate. This is known as the **washer method**:

$$V = \pi \int_a^b (\text{outer radius})^2 - (\text{inner radius})^2 \, dx \,. \tag{2}$$

Exercise 2: Suppose that the region between $y = x^2$ and $y = x$ is rotated about the line $y = 1$. Find the volume of the resulting solid of revolution (see Figure 3).

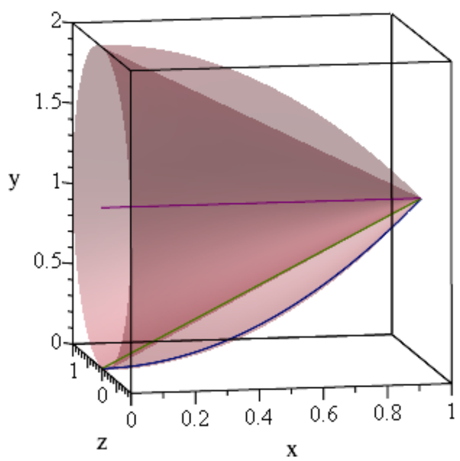


Figure 3: The solid of revolution obtained by rotating the region between $y = x^2$ and $y = x$ about the line $y = 1$. The solid in question can be found by removing the inner cone (dark red) from the outer bowl (light red).

Rotating about the y-axis

Exercise 3: Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \frac{1}{2}x^2, x = 0$, and $y = 3$ about the y-axis. Sketch the solid.

Hint: Use the disc method, but with cross sections perpendicular to the y-axis. You should integrate with respect to y.

Exercise 4: Find the volume of the solid of revolution obtained by rotating the region bounded by $y = x^2$ and $y = x$ about the y-axis (see Figure 4). Is this solid bigger or smaller than the solid from Exercise 2?

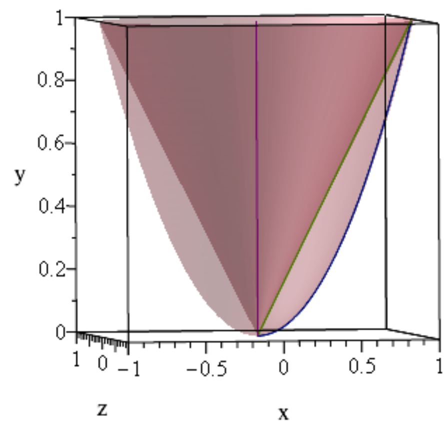


Figure 4: The solid of revolution obtained by rotating the region between $y = x^2$ and $y = x$ about the y-axis. The solid in question can be found by removing the inner cone (dark red) from the outer bowl (light red).