MATH 134 Calculus 2 with FUNdamentals

Section 5.7: u-Substitution

Recall the chain rule:

$$\frac{d}{dx}\left[f(u(x))\right] \;=\; f'(u(x)) \cdot u'(x).$$

If we take the integral of both sides, we find

$$\int f'(u)\,du \;=\; \int f'(u(x)) \cdot u'(x)\,dx \;=\; \int \frac{d}{dx}\left[f(u(x))\right]\,dx \;=\; f(u(x))\,.$$

This suggests a technique for finding an antiderivative: determine the "inside" function $u(x)$ and make a substitution, called a **u-sub** for short, that turns the integrand into a simpler integral in the variable u. The technique of **u-substitution** essentially uses the chain-rule backwards.

Example 1: Evaluate $\int 6x(3x^2+7)^8\,dx$ using the substitution $u = 3x^2+7$.

Answer: Since $u = 3x^2+7$, $\frac{du}{dx} = 6x$ or $du = 6x\,dx$. Making this substitution, the integral transforms into

$$\int u^8\,du \;=\; \frac{1}{9}u^9 + c \;=\; \frac{1}{9}(3x^2+7)^9 + c.$$

Note that we return to the variable x at the end of the problem. We can easily check our answer by using the chain rule:

$$\frac{d}{dx}\left[\frac{1}{9}(3x^2+7)^9 + c\right] \;=\; (3x^2+7)^8 \cdot 6x \;=\; 6x(3x^2+7)^8,$$

as desired. Here is another example.

Example 2: Evaluate $\int xe^{-x^2}\,dx$ using the substitution $u = -x^2$.

Answer: Here we have $u = -x^2$ so that $\frac{du}{dx} = -2x$ or $du = -2x\,dx$. While we have an $x\,dx$ term in the integrand, we are missing a factor of -2. We use the simple trick of multiplying and dividing the integrand by -2 to make it look the way we want, remembering that constants pull out of integrals.

$$\int xe^{-x^2}\,dx \;=\; \int -\frac{1}{2}\cdot -2xe^{-x^2}\,dx \;=\; -\frac{1}{2}\int -2xe^{-x^2}\,dx \;=\; -\frac{1}{2}\int e^u\,du \;=\; -\frac{1}{2}e^u + c \;=\; -\frac{1}{2}e^{-x^2} + c.$$

Alternatively, we could have solved $\frac{du}{dx} = -2x$ for dx, obtaining $dx = \frac{du}{-2x}$ and then made the substitution. Either way, we obtain the same integral in the variable u. The key to the u-sub technique of integration is to find the correct substitution u and then transform the integral into an easier one that **only involves the variable u.**

Our third example explains how to use u-substitution with a definite integral.

Example 3: Evaluate $\int_0^1 x^2(1+2x^3)^5\,dx$ using the substitution $u = 1+2x^3$.

Answer: Since $u = 1+2x^3$, we have $du = 6x^2\,dx$ so we need to multiply the integrand by 6 and pull out the constant $\frac{1}{6}$. But we also need to change the limits of integration. If $x = 0$, then $u = 1+2(0)^3 = 1$ and if $x = 1$, then $u = 1+2(1)^3 = 3$. Thus, our definite integral becomes

$$\int_0^1 x^2(1+2x^3)^5\,dx \;=\; \int_0^1 \frac{1}{6}\cdot 6x^2(1+2x^3)^5\,dx \;=\; \frac{1}{6}\int_1^3 u^5\,du \;=\; \frac{1}{36}u^6\Big|_1^3 \;=\; \frac{1}{36}(3^6-1) \;=\; \frac{182}{9}.$$

Exercises:

1. Evaluate $\int 12x^3\sqrt{3x^4+1}\,dx$ using the substitution $u = 3x^4+1$.

2. Evaluate $\int \frac{4t}{t^2+1}\,dt$ using the substitution $u = t^2+1$.

3. Evaluate $\int \tan\theta\,d\theta$ using the substitution $u = \cos\theta$. Why won't the substitution $u = \sin\theta$ work?

4. Evaluate $\int (x-2)\sqrt{x+1}\,dx$ using the substitution $u = x+1$.

   Hint: Solve for x in order to convert the integrand into a function of only u.

5. Evaluate $\int \frac{\cos x + 4}{(\sin x + 4x)^3}\,dx$.

6. Evaluate $\int_1^e \frac{\ln x}{x}\,dx$.

7. Evaluate $\int_0^{\pi/4} \sin^3(2\theta)\cos(2\theta)\,d\theta$.

8. Evaluate $\int_{-5}^{5} \frac{x^5-3x^3+7x}{x^6+4}\,dx$.