

# MATH 134 Calculus 2 with FUNdamentals

## Section 5.8: Further Transcendental Functions

This section focuses on a few more integration formulas involving exponential and inverse trig functions. Here, the challenging aspect is not the new formulas, but rather making the correct  $u$ -substitution in order to convert the integrand into the required form.

The new integral formulas come from the corresponding formulas for the derivative:

$$1. \int a^x dx = \frac{a^x}{\ln a} + c \quad \text{for any real number } a > 0$$

$$2. \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$

$$3. \int \frac{1}{x^2+1} dx = \tan^{-1} x + c$$

Note that in formula 1., if we choose  $a = e$ , then we obtain  $\int e^x dx = e^x + c$  (as expected) because  $\ln e = 1$ .

**Example 1:** Evaluate  $\int_0^{1/2} \frac{3}{\sqrt{1-x^2}} dx$ .

**Answer:** Using formula 2., we have

$$\int_0^{1/2} \frac{3}{\sqrt{1-x^2}} dx = 3 \sin^{-1}(x) \Big|_0^{1/2} = 3(\sin^{-1}(1/2) - \sin^{-1}(0)) = 3(\pi/6 - 0) = \pi/2.$$

**Example 2:** Find  $\int \frac{1}{9x^2+16} dx$  using the substitution  $u = \frac{3x}{4}$ .

**Answer:** The hard part here is that we cannot directly apply formula 3. because the denominator isn't in the correct form. The trick is to make a  $u$ -substitution so that the new integral can be evaluated with formula 3. Letting  $u = \frac{3x}{4}$ , we have  $du = \frac{3}{4} dx$  or  $dx = \frac{4}{3} du$ . We also have

$$x = \frac{4u}{3} \implies 9x^2 + 16 = 9\frac{16u^2}{9} + 16 = 16(u^2 + 1).$$

Thus, in the  $u$ -variable, the integral transforms to

$$\int \frac{1}{16(u^2+1)} \cdot \frac{4}{3} du = \frac{1}{12} \int \frac{1}{u^2+1} du = \frac{1}{12} \tan^{-1} u + c = \frac{1}{12} \tan^{-1} \left( \frac{3x}{4} \right) + c.$$

In general, the following  $u$ -substitutions allow for successful applications of formulas 2 and 3:

- For integrals of the form  $\int \frac{1}{\sqrt{a^2 - b^2x^2}} dx$ , use  $u = \frac{bx}{a}$ .
- For integrals of the form  $\int \frac{1}{a^2x^2 + b^2} dx$ , use  $u = \frac{ax}{b}$ .

**Exercises:**

1. Evaluate  $\int_0^1 \frac{4}{x^2 + 1} dx$ .

2. Evaluate  $\int_0^{1/2} 9^t dt$

3. Evaluate  $\int \frac{1}{1 + 36x^2} dx$  using the substitution  $u = 6x$ .

4. Evaluate  $\int_0^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9 - 4x^2}}$  using the substitution  $u = \frac{2x}{3}$ .

5. Evaluate  $\int x^2 5^{x^3} dx$ .

6. Evaluate  $\int_0^{\sqrt{3}} \frac{x + 1}{x^2 + 1} dx$ . **Hint:** Split the integrand into two pieces.