MATH 134 Calculus 2 with FUNdamentals

Section 5.5: The FUNdamental Theorem of Calculus, Part 2

This worksheet focuses on the second (and more difficult) part of the Fundamental Theorem of Calculus (FTC). In essence, it states that differentiation and integration are inverse processes. Part 2 can be used to give a simple proof of Part 1 of the FTC.

The Fundamental Theorem of Calculus (FTC), Part 2

Suppose that f is a continuous function over a closed interval $a \le x \le b$. If

$$A(x) \;=\; \int_a^x f(t)\,dt$$

is the **area function** giving the area under f from a to x, then $A(x)$ is differentiable and its derivative is just $f(x)$. In other words, $A'(x) = f(x)$ or

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) \;=\; f(x). \qquad (1)$$

Some Important Notes Concerning Part 2 of the FTC:

(i) In part 2 of the FTC, the variable is x and it is the upper limit of integration. The lower limit a is an arbitrary constant. As x varies, the area under the curve varies, which is why $A(x)$ is really a function. The amazing aspect of part 2 is that this function has a derivative equal to the function we are finding the area under, namely f.

(ii) A simple way to remember part 2, and in particular equation (1), is that the derivative and integral sign cancel out. In other words, differentiation and integration are inverse processes — doing one operation and then the other gets you back to where you started. But be careful, the variable x must be a **limit of integration!** Check out one of my favorite final exam questions below:

Exercise 0: Evaluate

$$\frac{d}{dx}\left(\int_0^7 \sin(e^{t^2})\,dt\right)$$

The answer is NOT $\sin(e^{x^2})$. What is it?

Exercises:

1. **Warm-up:** If $A(x) = \displaystyle\int_1^x t^2\,dt$, show that $A'(x) = x^2$ by first evaluating the integral and then taking the derivative.

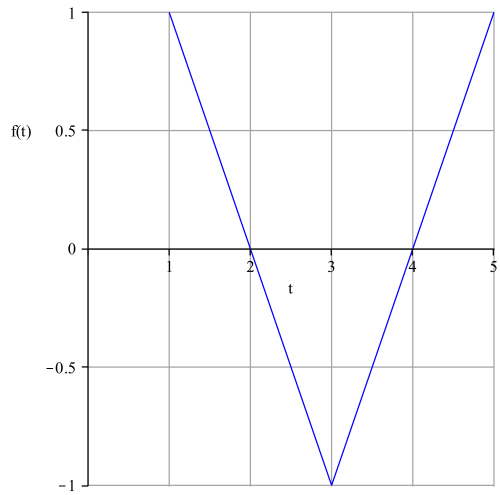
2. Find $\dfrac{d}{dx}\left(\displaystyle\int_8^x e^{t^2}\,dt\right)$, and then find $\dfrac{d}{dx}\left(\displaystyle\int_x^8 e^{t^2}\,dt\right)$.

3. Find $\dfrac{d}{dx}\left(\displaystyle\int_4^{\sqrt{x}} \sin(t^2)\,dt\right)$. **Hint:** Use the chain rule where the "inside" function is $\sqrt{x}$.

4. If $G(x) = \displaystyle\int_{x^3}^{5} \sqrt{t^5+5}\,dt$, find $G'(x)$.

5. If $H(x) = \displaystyle\int_{x^3}^{e^x} \sqrt{t^5+5}\,dt$, find $H'(x)$. **Hint:** Break the integral into two integrals. Then differentiate.

6. Suppose that $G(x) = \int_1^x f(t)\,dt$, where the graph of f is shown below.



(a) Find $G(1), G(2), G(3), G(4)$ and $G(5)$.

(b) Find $G'(2), G'(3)$ and $G'(5)$.

(c) Where is $G(x)$ concave up? Does $G''(3)$ exist? Explain.

(d) Sketch the graph of G over the interval $1 \le x \le 5$.

(e) **Challenge:** Find a formula for $G(x)$. **Hint:** G is a piecewise function.