MATH 134 Calculus 2 with FUNdamentals

Section 5.4: The FUNdamental Theorem of Calculus, Part 1

This section focuses on the most important theorem in calculus. In fact, the Fundamental Theorem of Calculus (FTC) is arguably one of the most important theorems in all of mathematics. In essence, it states that differentiation and integration are inverse processes. There are two parts to the FTC; this worksheet focuses on the first part.

The Fundamental Theorem of Calculus (FTC), Part 1

Suppose that $f(x)$ is a continuous function over a closed interval $a \leq x \leq b$. Then

$$\int_a^b f(x)dx \;=\; F(b) - F(a), \quad \text{where } F \text{ is any antiderivative of } f.$$

Some Important Notes Concerning Part 1 of the FTC:

(i) The first part of the FTC will be used over and over again throughout the course. It states that to find the signed area under f from a to b, we need only find an antiderivative of f, evaluate it at the endpoints a and b, and then subtract. There is something fundamentally surprising going on here: somehow, the area only depends on the values of the antiderivative at the endpoints, not in between. How is this possible? It is a deep fact and one that has a useful generalization into higher dimensions via an important theorem known as *Stokes Theorem.*

(ii) Since F is an antiderivative of F' (by definition), another way to write part 1 of the FTC is

$$\int_a^b F'(t)\,dt \;=\; F(b) - F(a). \tag{1}$$

Notice the integral sign and derivative sign canceling out here. Equation (1) basically says that integrating a rate of change gives the net (or total) change. An example from economics is

$$\int_a^b C'(x)\,dx \;=\; C(b) - C(a),$$

which states that the integral of the **marginal cost** gives the total change in cost when increasing the number of items produced from a to b units. In physics, recall that velocity $v(t)$ is the derivative of position $s(t)$. This implies

$$\int_a^b s'(t)\,dt \;=\; \int_a^b v(t)\,dt \;=\; s(b) - s(a).$$

In words, the integral of velocity over $a \leq t \leq b$ gives the total change in position (sometimes called the net displacement). We already saw this application in Section 5.1 in terms of areas.

(iii) It is particularly important to understand the difference between $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$. The limits of integration, or lack thereof, are critical. The $\int f(x)\,dx$ is an **indefinite integral** and represents the general antiderivative of $f(x)$. Here you can think of the integral sign as telling you to find the general antiderivative. For example, we have

$$\int x^2\,dx \;=\; \frac{1}{3}x^3 + c \quad \text{and} \quad \int \cos x\,dx \;=\; \sin x + c\,.$$

The quantity $\int_a^b f(x)\,dx$ is the **definite integral**, and represents the signed area under f from a to b. The definite integral is a **number**, while the indefinite integral is a **function.**

Exercises:

1. Use the FTC to compute $\displaystyle\int_0^3 x+2\,dx$ and $\displaystyle\int_{-3}^3 x+2\,dx$.

   Check that your answers agree with those you obtained for Exercises 2 and 3 on the worksheet for Section 5.2 (The Definite Integral).

2. Use the FTC to compute $\displaystyle\int_0^\pi \sin\theta\,d\theta$ and $\displaystyle\int_0^{2\pi} \sin\theta\,d\theta$.

   Interpret your answers graphically in terms of signed area.

3. Evaluate each definite integral.

   (a) $\displaystyle\int_0^1 10e^{5t}\,dt$

   (b) $\displaystyle\int_1^9 \frac{1}{x} - \frac{3}{\sqrt{x}}\,dx$

   (c) $\displaystyle\int_{\pi/3}^{\pi/2} 2\sin(2x) + 6\cos(3x)\,dx$

   (d) Challenge Problem: $\displaystyle\int_0^4 |x^2 - 4x + 3|\,dx$ **Hint:** Draw a graph of the integrand and then break the integral up into three pieces without any absolute value signs.