MATH 134 Calculus 2 with FUNdamentals

Section 5.2: The Definite Integral

In this section we define the **definite integral**, which represents the signed area under the graph of a function. The value of the integral may be positive, negative, or zero depending on whether the function is above or below the x-axis (or perhaps both).

Definition: If f is a continuous function on $a \leq x \leq b$, and $\Delta x = \frac{b-a}{n}$, where n corresponds to the number of subdivisions of $[a, b]$, then the **definite integral of f from a to b** is

$$\int_a^b f(x)\, dx \;=\; \lim_{n\to\infty} (f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x)$$

where x_i^* is any sample point in the subinterval $[x_i, x_{i+1}]$.

The $\int$ symbol is an elongated S standing for "summation." The constants a and b are called the **limits of integration** and the function inside the integral, $f(x)$, is called the **integrand**.

The summation in the parentheses above is called a **Riemann Sum**. If for each i, x_i^* is the left endpoint of the ith subinterval, then the resulting sum is a **Left-hand Sum**, abbreviated L_n. Alternatively, if for each i, x_i^* is the right endpoint of the ith subinterval, then the resulting sum is a **Right-hand Sum**, abbreviated R_n. Finally, if for each i, x_i^* is the midpoint of the ith subinterval, then the resulting sum is a **Midpoint Sum**, abbreviated M_n. But for a general Riemann Sum, x_i^* can be **any** point in the given subinterval.

As n gets larger, the width of the approximating rectangles goes to zero; however, the number of such rectangles is going to infinity. If f is continuous, this limit always exists as a finite number and represents the **signed area** under the curve. This means, for example, that if the graph of f lies entirely below the axis, then the definite integral will be negative. Thus, "negative area" is allowed.

Some Important Notes:

- The definite integral is a **number**!
- Treating $\int_a^b f(x)\, dx$ as signed area, the area of the parts of f above the axis count as positive area while the area of the parts of f below the axis count as negative area. The total signed area is the sum of these two numbers.
- The variable inside the integral is irrelevant, and is usually called a **dummy variable**. Thus,

$$\int_a^b f(x)\, dx \;=\; \int_a^b f(t)\, dt \;=\; \int_a^b f(\theta)\, d\theta \;=\; \int_a^b f(H_C)\, dH_C.$$

  What matters is f and the limits of integration, not the particular variable. It is typical to think of dx as indicating to integrate with respect to x.

Exercise 1: Explain why $\displaystyle\int_0^{2\pi} \sin x\, dx = 0$ by drawing a graph of $y = \sin x$ over the interval $[0, 2\pi]$.

Is it also true that $\displaystyle\int_0^{2\pi} \cos x\, dx = 0$?

Since we already know how to compute the areas of certain familiar geometric shapes (e.g., triangles, rectangles, circles), we can evaluate certain definite integrals by graphing the integrand and calculating the area under the curve using a well-known area formula.

Exercise 2: Calculate $\int_0^3 x + 2 \, dx$ by interpreting the definite integral in terms of areas.

Exercise 3: Calculate $\int_{-3}^3 x + 2 \, dx$ by interpreting the definite integral in terms of areas.

Exercise 4: Calculate $\int_{-4}^4 \sqrt{16 - x^2} \, dx$ by interpreting the definite integral in terms of areas.

Hint: Let $y = \sqrt{16 - x^2}$. Square both sides and put all the variables on one side of the equation. What figure does the equation represent?

Below are some important properties of the definite integral. Most of these follow from the definition at the top of page 1 and properties of limits.

Properties of Integrals

1. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (interchanging the limits of integration pulls out a minus sign)

2. $\displaystyle\int_a^a f(x)\,dx = 0$

3. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ (linearity of integration limits)

4. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (linearity of the integral)

5. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ (constants pull out)

6. $\displaystyle\int_a^b m\,dx = m(b-a)$ (area of a rectangle)

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ (comparing integrals)

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Additional Exercises:

5. If $\displaystyle\int_1^3 f(x)\,dx = 5$ and $\displaystyle\int_3^7 f(x)\,dx = -2$, find $\displaystyle\int_1^7 8f(x)\,dx$.

6. If $\displaystyle\int_1^8 f(x)\,dx = 7$ and $\displaystyle\int_4^8 f(x)\,dx = 10$, find $\displaystyle\int_1^4 \pi f(x)\,dx$.

7. Evaluate $\int_0^3 5 - 2\sqrt{9 - t^2}\, dt$.

8. Evaluate $\int_{-2}^5 |x - 1|\, dx$. *Hint:* Graph $|x - 1|$ carefully.

9. Evaluate $\int_{-20}^{20} \sin\theta + \theta^{55} - 8\theta\, d\theta$. *Hint:* What type of function is the integrand?