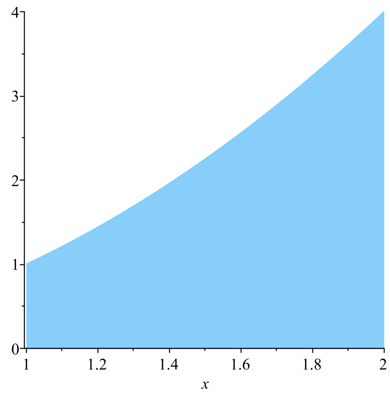
MATH 134 Calculus 2 with FUNdamentals

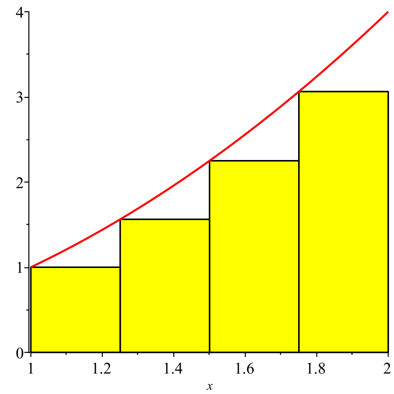
Section 5.1: Approximating and Computing Area

We begin the course with a review of the first few sections of Chapter 5, *The Integral*. The focus of this chapter is on computing the area under the graph of a function. Recall the amazing fact that differentiation (i.e., finding the slope of the tangent line) and integration (i.e., finding the area under a curve) are related in a special way: they are inverse processes.

Approximating Area by Rectangles

In this section we learn a simple technique for approximating the area under a curve by using rectangles. There are different ways to choose the heights of the rectangles that lead to different types of sums. The basic idea is that the more rectangles we use, the better the approximation becomes.





Example 1: Approximate the area under the graph of $f(x) = x^2$ between $x = 1$ and $x = 2$ using four rectangles. Specifically, compute the Left-hand Sum L_4, the Right-hand Sum R_4, and the Midpoint Sum M_4.

Answer: The goal is to approximate the area under the parabola $y = f(x) = x^2$, above the x-axis, and between the vertical lines $x = 1$ and $x = 2$ (the blue shaded region in the figure above). We are told to use four rectangles. The simplest approach is to choose rectangles with the same width Δx. Since the length of the interval $[1, 2]$ is 1, we let $\Delta x = 1/4$ be the width of each rectangle. Divide the x-axis between 1 and 2 into four equal subintervals, each of length $1/4 = 0.25$:

$$[1, 1.25], \ [1.25, 1.5], \ [1.5, 1.75], \quad \text{and} \quad [1.75, 2].$$

Next, we will determine the height of each rectangle by evaluating the function at the **left endpoints** of each subinterval. This is known as a **Left-hand Sum**, denoted L_n (where n is the number of rectangles). In this case, we are computing L_4 (see the right-hand figure above). Notice that the height of each rectangle is found by the value of the function at the left endpoint of each subinterval. We have

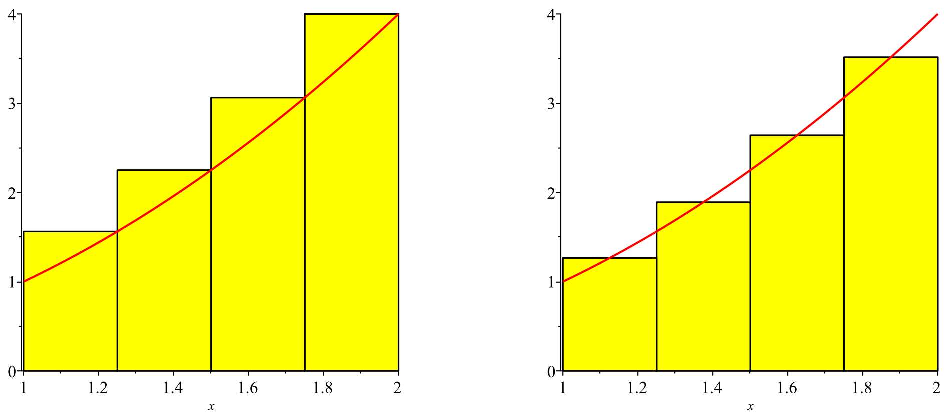
$$\begin{aligned} L_4 &= \frac{1}{4} \cdot f(1) + \frac{1}{4} \cdot f(1.25) + \frac{1}{4} \cdot f(1.5) + \frac{1}{4} \cdot f(1.75) \\ &= \frac{1}{4} \left(f(1) + f(1.25) + f(1.5) + f(1.75) \right) \\ &= \frac{1}{4} \left(1^2 + 1.25^2 + 1.5^2 + 1.75^2 \right) \\ &= 1.96875 \,. \end{aligned}$$

According to the graph of the Left-hand Sum, $L_4 = 1.96875$ is an **underestimate** of the actual area because each rectangle lies below the graph of the function.

To compute the **Right-hand Sum** R_4, we use the same width $\Delta x = 0.25$ and same four subintervals, but now we choose the **right endpoints** of each subinterval to calculate the height of each rectangle (see the left-hand figure below). This gives

$$\begin{aligned} R_4 &= \frac{1}{4} \cdot f(1.25) + \frac{1}{4} \cdot f(1.5) + \frac{1}{4} \cdot f(1.75) + \frac{1}{4} \cdot f(2) \\ &= \frac{1}{4} \left(f(1.25) + f(1.5) + f(1.75) + f(2) \right) \\ &= \frac{1}{4} \left(1.25^2 + 1.5^2 + 1.75^2 + 2^2 \right) \\ &= 2.71875\,. \end{aligned}$$

Based on the graph of the Right-hand Sum, $R_4 = 2.71875$ is an **overestimate** of the actual area because each rectangle lies above the graph of the function.



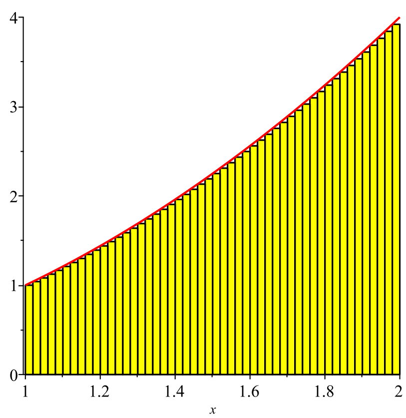
Finally, we compute our best approximation, the **Midpoint Sum** M_4 by using four rectangles of width $\Delta x = 0.25$ and heights given by the **midpoints** of each subinterval (see the right-hand figure above). Notice that the height of each rectangle is determined by the function value at the midpoints of each subinterval, $1.125, 1.375, 1.625$, and 1.875. Recall that the midpoint of an interval is found by taking the average of its endpoints. We compute

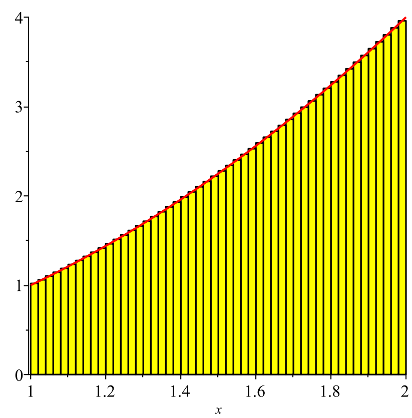
$$\begin{aligned} M_4 &= \frac{1}{4} \cdot f(1.125) + \frac{1}{4} \cdot f(1.375) + \frac{1}{4} \cdot f(1.625) + \frac{1}{4} \cdot f(1.875) \\ &= \frac{1}{4} \left(f(1.125) + f(1.375) + f(1.625) + f(1.875) \right) \\ &= \frac{1}{4} \left(1.125^2 + 1.375^2 + 1.625^2 + 1.875^2 \right) \\ &= 2.328125\,. \end{aligned}$$

The value $M_4 = 2.328125$ is the best approximation of the three because each rectangle contains a piece above the curve and misses a similar shaped piece below. The actual value of the area under the curve is $A = 7/3 = 2.333\ldots$. We will learn how to compute this value in Section 5.4.

Exercise 1: Approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 2$ by computing **(a)** L_{10} and **(b)** M_{10}. Do your approximations overestimate or underestimate the actual area under the graph?
Hint: Recall that $\Delta x = (b-a)/n$ gives the width obtained by dividing the interval $[a, b]$ into n pieces.

If you did the previous exercise correctly, you should find that L_{10} is a better approximation to the true area under the curve than L_4. In general, the more rectangles we use to approximate the area, the better the approximation becomes. Below are two approximations for $f(x) = x^2$ between $x = 1$ and $x = 2$ using $n = 50$ rectangles. The plot on the left is a Left-hand Sum with area $L_{50} = 2.3034$, while the right-hand plot is a Midpoint Sum with area $M_{50} = 2.3333$. Notice how much better these approximations are to the actual area of $2.333\ldots$.





Formulas for L_n, R_n, and M_n

Here are the formulas for the three different types of approximations. Suppose that the interval $[a, b]$ is divided into n equal subintervals of width $\Delta x = (b - a)/n$. Let the consecutive endpoints of the subintervals be denoted as $a = x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b$. (Notice that there are $n + 1$ total endpoints.) Then we have

$$\begin{aligned} L_n &= \Delta x \left[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) \right] \\ R_n &= \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right] \\ M_n &= \Delta x \left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right) \right] \end{aligned}$$

Notice that for L_n, we ignore the rightmost endpoint $b = x_n$; for R_n, we ignore the leftmost endpoint $a = x_0$; and for M_n, we average the endpoints on each subinterval *before* plugging into the function.

Exercise 2: Approximate the area under the graph of the function $f(x) = 1/x$ between $x = 2$ and $x = 5$ by computing **(a)** L_6, **(b)** R_6, and **(c)** M_6. Draw pictures of each sum and determine whether each sum is an overestimate or underestimate.

Application: Distance Traveled

Suppose that $v(t)$ represents the velocity of a moving object (e.g., a car). If we want to approximate the distance traveled over a given time, we multiply the velocity by the time. For example, if we drive 40 miles/hour for 2 hours, then we have traveled 80 miles ($40 \cdot 2 = 80$). This can be interpreted geometrically as the height of the function, $v(t) = 40$, times the width along the t-axis, $\Delta t = 2$. In other words, the distance traveled is just the area of a rectangle. Generalizing this concept, we have

Distance traveled $=$ area under the graph of the velocity function $v(t)$

This formula assumes that $v(t) > 0$. It will be needed on HW #1 and discussed in greater detail in Section 5.6.

We close this section by stating an important fact about the area under the graph of a continuous function. If we let the number of rectangles become arbitrarily large (i.e., let $n \to \infty$), then the Left-hand, Right-hand, and Midpoint Sums all limit on the same value A. We define this value to be the area under the curve.

Area Theorem: If $f(x)$ is a continuous function on the interval $[a, b]$, then

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n = A\,.$$

We define A to be the area under the graph of $f(x)$ between $x = a$ and $x = b$.