MATH 134 Calculus 2 with FUNdamentals

Section 10.3: Convergence of Series with Positive Terms

SOLUTIONS

Exercise 1: Use the integral test to determine whether the given series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Answer: **(a)** We start by defining $f(x) = 1/\sqrt{x} = x^{-1/2}$. This function is positive, decreasing, and continuous for $x \geq 1$. It is decreasing because $f'(x) = -\frac{1}{2}x^{-3/2} < 0$. Using the integral test, we compute

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx \;=\; \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx \;=\; \lim_{b\to\infty} 2x^{1/2}\Big|_1^b \;=\; \lim_{b\to\infty} 2\sqrt{b}-2 \;=\; \infty\,.$$

Since the improper integral diverges, the series also diverges.

(b) In this case, we let $f(x) = 1/(x^2+1) = (x^2+1)^{-1}$. This function is positive, decreasing, and continuous for $x \geq 1$. It is decreasing because $f'(x) = -2x(x^2+1)^{-2} < 0$. Using the integral test, we compute

$$\int_1^\infty \frac{1}{x^2+1}\,dx \;=\; \lim_{b\to\infty}\int_1^b \frac{1}{x^2+1}\,dx \;=\; \lim_{b\to\infty} \tan^{-1}(x)\Big|_1^b \;=\; \lim_{b\to\infty} \tan^{-1}(b) - \tan^{-1}(1) \;=\; \frac{\pi}{2}-\frac{\pi}{4} \;=\; \frac{\pi}{4}\,.$$

Since the improper integral converges, the series also converges.

Exercise 2: Using an appropriate test for convergence, determine whether the given infinite series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Answer: This series converges by the p-series test.

Since $n\sqrt{n} = n\cdot n^{1/2} = n^{3/2}$, the series can be written as $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$. This is a p series with $p = 3/2$.

Since $3/2 > 1$, the series converges by the p-series test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+4}$

Answer: This series diverges by the nth term test.

Using L'Hôpital's Rule, or by inspection, we have

$$\lim_{n\to\infty}\frac{n^2}{n^2+4} \;=\; \lim_{n\to\infty}\frac{2n}{2n} \;=\; 1 \;\neq\; 0\,.$$

Since $\displaystyle\lim_{n\to\infty} a_n \neq 0$, the series diverges by the nth term test.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Answer: This series diverges by the integral test.

We let $f(x) = 1/(x \ln x) = (x \ln x)^{-1}$. This function is positive, decreasing, and continuous for $x \geq 2$. It is decreasing because $f'(x) = -(x \ln x)^{-2}(\ln x + 1) < 0$. Using the integral test, we compute

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx \;=\; \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x}\, dx \;=\; \lim_{b \to \infty} \ln(\ln x)\Big|_2^b \;=\; \lim_{b \to \infty} \ln(\ln b) - \ln(\ln 2) \;=\; \infty\,.$$

The integral is evaluated by doing a u-substitution with $u = \ln x, du = \frac{1}{x}\, dx$. This transforms the integral into $\displaystyle\int \frac{1}{u}\, du = \ln u$. Since the improper integral diverges, the series also diverges.