MATH 134-02, Spring 2014

The FUNdamental Theorem of Calculus

This worksheet focuses on the most important theorem in calculus. In fact, the Fundamental Theorem of Calculus (FTC) is arguably one of the most important theorems in all of mathematics. In essence, it states that differentiation and integration are inverse processes. The problems on this sheet pertain to the first part of FTC because that is the hardest part to understand.

The Fundamental Theorem of Calculus (FTC)
Suppose that f is a continuous function over a closed interval $a \leq x \leq b$.

1. If $A(x) = \displaystyle\int_a^x f(t)\,dt$ is the area function giving the area under f from a to x, then $A(x)$ is differentiable and its derivative is just $f(x)$. In other words, $A'(x) = f(x)$ or

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x). \tag{1}$$

2. $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f.

Some Very Important Notes on the FTC:

(i) In part 1., the variable is x and it is the upper limit of integration. The lower limit a is an arbitrary constant. As x varies, the area under the curve varies, which is why $A(x)$ is really a function. The amazing fact demonstrated in part 1. is that this function has a derivative equal to the function we are finding the area under, namely f.

(ii) A simple way to remember part 1., and in particular equation (1), is that the derivative and integral sign cancel out. In other words, differentiation and integration are inverse processes — doing one operation and then the other gets you back to where you started. But be careful, the variable x must be a **limit of integration!** Check out one of my favorite final exam questions below:

Exercise 0: Calculate

$$\frac{d}{dx}\left(\int_0^7 \sin(e^{t^2})\,dt\right)$$

The answer is NOT $\sin(e^{x^2})$.

(iii) It is particularly important to understand the difference between $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$. The limits of integration, or lack thereof, are critical. $\int f(x)\,dx$ is called an **indefinite integral** and represents the general antiderivative of $f(x)$. Here you can think of the integral sign as telling you to find the general antiderivative. For example, we have

$$\int x^2\,dx = \frac{1}{3}x^3 + c \quad \text{and} \quad \int \cos x\,dx = \sin x + c.$$

The quantity $\int_a^b f(x)\,dx$ is called the **definite integral**, and represents the area under f from a to b. The definite integral is a **number**!

(iv) The second part of the FTC will be used over and over again throughout the course. It states that to find the area under f from a to b, we need only find an antiderivative of f, evaluate it at the endpoints a and b, and then subtract. There is something fundamentally surprising going on here: somehow, the area only depends on the values of the antiderivative at the endpoints, not in between. How is this possible? It is a deep fact and one that has a useful generalization into higher dimensions via a famous theorem known as *Stokes Theorem.*

(v) Since F is an antiderivative of F' (by definition), another way to write part 2. of the FTC is

$$\int_a^b F'(t)\, dt \;=\; F(b) - F(a). \tag{2}$$

Again, notice the integral sign and derivative sign canceling out here. Equation (2) basically says that integrating a rate of change gives the **net** (or total) change. We have already seen this in terms of Physics with the relationship between position $s(t)$ and velocity $v(t)$:

$$\int_a^b s'(t)\, dt \;=\; \int_a^b v(t)\, dt \;=\; s(b) - s(a).$$

In words, the integral of velocity over $a \le t \le b$ gives the total change in position (sometimes called the net displacement). An example from Economics would be something like

$$\int_a^b C'(x)\, dx \;=\; C(b) - C(a),$$

which states that the integral of the **marginal cost** gives the total change in cost when increasing the number of items produced from a to b units.

(vi) The variable inside a definite integral is irrelevant, and is usually called a **dummy variable**. Thus,

$$\int_a^b f(x)\, dx \;=\; \int_a^b f(t)\, dt \;=\; \int_a^b f(\theta)\, d\theta \;=\; \int_a^b f(H_C)\, dH_C.$$

What matters is f and the limits of integration, not the particular variable. However, it is typical to think of dx as indicating to integrate with respect to x.

Exercises:

1. If $F(x) = \displaystyle\int_{-3}^{x} \sqrt[3]{\cos^2(3t) + 5}\, dt$, find $F'(x)$.

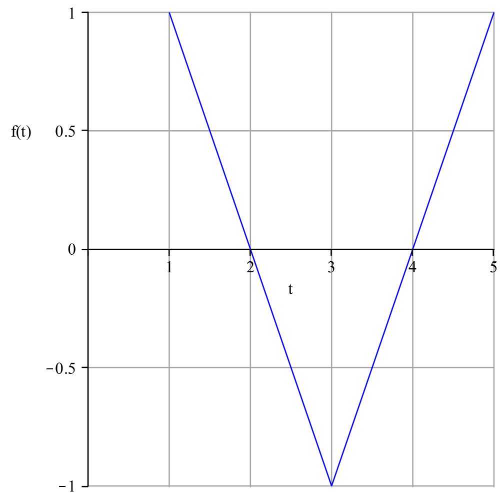
2. Find $\dfrac{d}{dx}\left(\displaystyle\int_8^x e^{t^2}\, dt\right)$, and then find $\dfrac{d}{dx}\left(\displaystyle\int_x^8 e^{t^2}\, dt\right)$.

3. Find $\dfrac{d}{dx}\left(\displaystyle\int_4^{\sqrt{x}} \sin(t^2)\,dt\right)$. *Hint:* Use the chain rule where the "inside" function is $\sqrt{x}$.

4. If $G(x) = \displaystyle\int_{x^3}^{5} \sqrt{t^5+5}\,dt$, find $G'(x)$.

5. If $H(x) = \displaystyle\int_{x^3}^{e^x} \sqrt{t^5+5}\,dt$, find $H'(x)$. *Hint:* Break the integral into two integrals. Then differentiate.

6. Suppose that $G(x) = \int_1^x f(t)\, dt$, where the graph of f is shown below.



(a) Find $G(1), G(2), G(3), G(4)$ and $G(5)$.

(b) Find $G'(2), G'(3)$ and $G'(5)$.

(c) Where is $G(x)$ concave up? Does $G''(3)$ exist? Explain.

(d) Sketch the graph of G over the interval $1 \le x \le 5$.