MATH 133 Calculus 1 with FUNdamentals

Section 5.4: The FUNdamental Theorem of Calculus

The last section of the course focuses on the most important theorem in calculus. In fact, the Fundamental Theorem of Calculus (FTC) is arguably one of the most important theorems in all of mathematics. In essence, it states that differentiation and integration are inverse processes. There are two parts to the FTC, but we will only focus on the first part (the second part will be taken up in *Calculus 2*).

The Fundamental Theorem of Calculus (FTC)

Suppose that $f(x)$ is a continuous function over a closed interval $a \leq x \leq b$. Then

$$\int_a^b f(x)dx \;=\; F(b) - F(a), \quad \text{where } F \text{ is any antiderivative of } f.$$

Some Important Notes Concerning the FTC:

(i) The FTC states that to find the signed area under f from a to b, we need only find an antiderivative of f, evaluate it at the endpoints a and b, and then subtract. There is something fundamentally surprising going on here: somehow, the area only depends on the values of the antiderivative at the endpoints, not in between. How is this possible? It is a deep fact and one that has a useful generalization into higher dimensions via an important theorem known as *Stokes Theorem.*

(ii) Since F is an antiderivative of F' (by definition), another way to write part 1 of the FTC is

$$\int_a^b F'(t)\,dt \;=\; F(b) - F(a). \tag{1}$$

Notice the integral sign and derivative sign canceling out here. Equation (1) basically says that integrating a rate of change gives the net (or total) change. An example from economics is

$$\int_a^b C'(x)\,dx \;=\; C(b) - C(a),$$

which states that the integral of the **marginal cost** gives the total change in cost when increasing the number of items produced from a to b units.

(iii) It is particularly important to understand the difference between $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$. The limits of integration, or lack thereof, are critical. The $\int f(x)\,dx$ is an **indefinite integral** and represents the general antiderivative of $f(x)$. Here you can think of the integral sign as telling you to find the general antiderivative. For example, we have

$$\int x^2\,dx \;=\; \frac{1}{3}x^3 + c \quad \text{and} \quad \int \cos x\,dx \;=\; \sin x + c\,.$$

The quantity $\int_a^b f(x)\,dx$ is the **definite integral**, and represents the signed area under f from a to b. The definite integral is a **number**, while the indefinite integral is a **function.**

Exercises:

1. Use the FTC to compute $\displaystyle\int_0^2 2x + 1\, dx$.

   Check that your answer agrees with the one you obtained for Exercise 2 on the worksheet for Section 5.2 (The Definite Integral).

2. Use the FTC to compute $\displaystyle\int_0^{\pi} \sin\theta\, d\theta$ and $\displaystyle\int_0^{2\pi} \sin\theta\, d\theta$.

   Interpret your answers graphically in terms of signed area.

3. Evaluate each definite integral.

   (a) $\displaystyle\int_0^1 6e^{3t} + \sqrt{t}\, dt$

   (b) $\displaystyle\int_1^5 \frac{1}{x} - \frac{5}{x^2}\, dx$

   (c) $\displaystyle\int_{\pi/3}^{\pi/2} \sin(3x) + 4\cos(2x)\, dx$