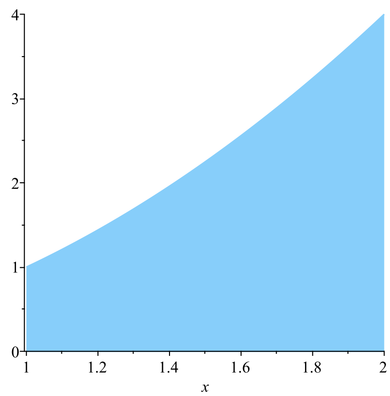
MATH 133 Calculus 1 with FUNdamentals

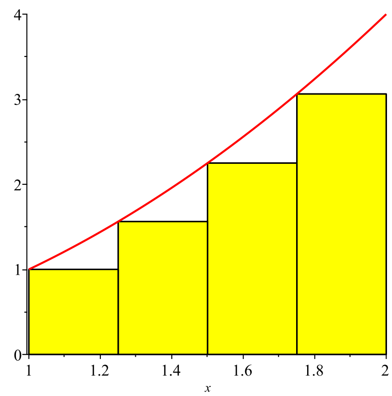
Section 5.1: Approximating and Computing Area

The final chapter we will study for the course is Chapter 5, *The Integral*. The focus of this chapter is on computing the area under the graph of a function. Remarkably, we will see that differentiation (i.e., finding the slope of the tangent line) and integration (i.e., finding the area under a curve) are related in a special way: they are inverse processes.

Approximating Area by Rectangles

In this section we learn a simple technique for approximating the area under a curve by using rectangles. There are different ways to choose the heights of the rectangles that lead to different types of sums. The basic idea is that the more rectangles we use, the better the approximation becomes.





Example 1: Approximate the area under the graph of $f(x) = x^2$ between $x = 1$ and $x = 2$ using four rectangles. Specifically, compute the Left-hand Sum L_4, the Right-hand Sum R_4, and the Midpoint Sum M_4.

Answer: The goal is to approximate the area under the parabola $y = f(x) = x^2$, above the x-axis, and between the vertical lines $x = 1$ and $x = 2$ (the blue shaded region in the figure above). We are told to use four rectangles. The simplest approach is to choose rectangles with the same width Δx. Since the length of the interval $[1, 2]$ is 1, we let $\Delta x = 1/4$ be the width of each rectangle. Divide the x-axis between 1 and 2 into four equal subintervals, each of length $1/4 = 0.25$:

$$[1, 1.25], \ [1.25, 1.5], \ [1.5, 1.75], \quad \text{and} \quad [1.75, 2].$$

Next, we will determine the height of each rectangle by evaluating the function at the **left endpoints** of each subinterval. This is known as a **Left-hand Sum**, denoted L_n (where n is the number of rectangles). In this case, we are computing L_4 (see the right-hand figure above). Notice that the height of each rectangle is found by the value of the function at the left endpoint of each subinterval. We have

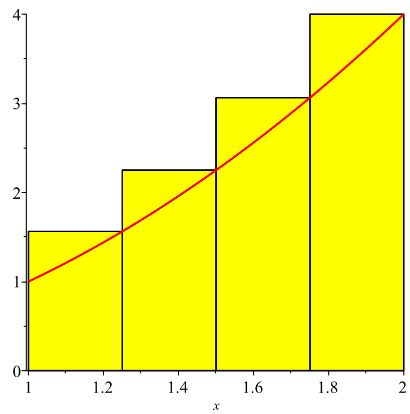
$$\begin{aligned} L_4 &= \frac{1}{4} \cdot f(1) + \frac{1}{4} \cdot f(1.25) + \frac{1}{4} \cdot f(1.5) + \frac{1}{4} \cdot f(1.75) \\ &= \frac{1}{4} \left(f(1) + f(1.25) + f(1.5) + f(1.75) \right) \\ &= \frac{1}{4} \left(1^2 + 1.25^2 + 1.5^2 + 1.75^2 \right) \\ &= 1.96875\,. \end{aligned}$$

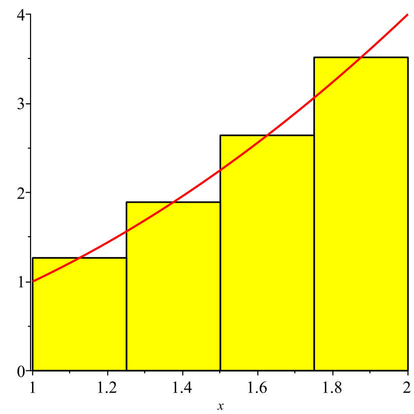
According to the graph of the Left-hand Sum, $L_4 = 1.96875$ is an **underestimate** of the actual area because each rectangle lies below the graph of the function.

To compute the **Right-hand Sum** R_4, we use the same width $\Delta x = 0.25$ and same four subintervals, but now we choose the **right endpoints** of each subinterval to calculate the height of each rectangle (see the left-hand figure below). This gives

$$\begin{aligned} R_4 &= \frac{1}{4} \cdot f(1.25) + \frac{1}{4} \cdot f(1.5) + \frac{1}{4} \cdot f(1.75) + \frac{1}{4} \cdot f(2) \\ &= \frac{1}{4} \left(f(1.25) + f(1.5) + f(1.75) + f(2) \right) \\ &= \frac{1}{4} \left(1.25^2 + 1.5^2 + 1.75^2 + 2^2 \right) \\ &= 2.71875\,. \end{aligned}$$

Based on the graph of the Right-hand Sum, $R_4 = 2.71875$ is an **overestimate** of the actual area because each rectangle lies above the graph of the function.





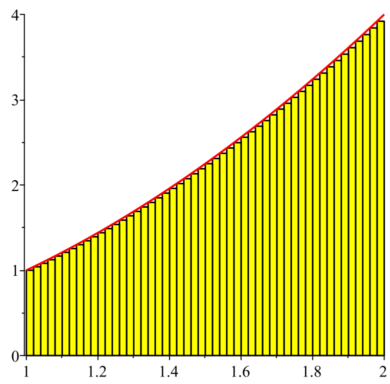
Finally, we compute our best approximation, the **Midpoint Sum** M_4 by using four rectangles of width $\Delta x = 0.25$ and heights given by the **midpoints** of each subinterval (see the right-hand figure above). Notice that the height of each rectangle is determined by the function value at the midpoints of each subinterval, $1.125, 1.375, 1.625$, and 1.875. Recall that the midpoint of an interval is found by taking the average of its endpoints. We compute

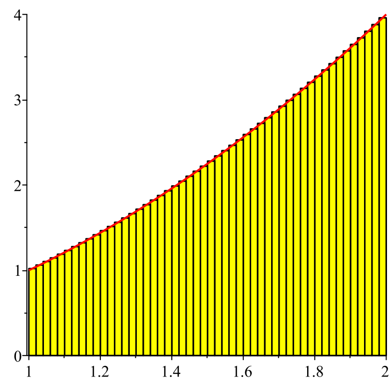
$$\begin{aligned} M_4 &= \frac{1}{4} \cdot f(1.125) + \frac{1}{4} \cdot f(1.375) + \frac{1}{4} \cdot f(1.625) + \frac{1}{4} \cdot f(1.875) \\ &= \frac{1}{4} \left(f(1.125) + f(1.375) + f(1.625) + f(1.875) \right) \\ &= \frac{1}{4} \left(1.125^2 + 1.375^2 + 1.625^2 + 1.875^2 \right) \\ &= 2.328125\,. \end{aligned}$$

The value $M_4 = 2.328125$ is the best approximation of the three because each rectangle contains a piece above the curve and misses a similar shaped piece below. The actual value of the area under the curve is $A = 7/3 = 2.333\ldots$. We will learn how to compute this value in Section 5.4.

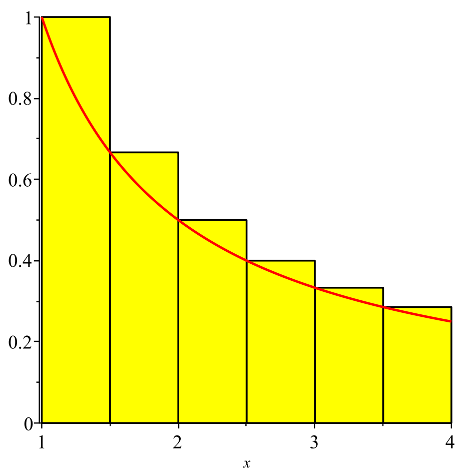
Exercise 1: Approximate the area under the function $f(x) = x^2$ between $x = 1$ and $x = 2$ by computing R_{10}. (Use right endpoints and $n = 10$ rectangles.) Is your approximation an overestimate or underestimate?

If you did the previous exercise correctly, you should find that R_{10} is a better approximation to the true area under the curve than R_4. In general, the more rectangles we use to approximate the area, the better the approximation becomes. Below are two approximations for $f(x) = x^2$ between $x = 1$ and $x = 2$ using $n = 50$ rectangles. The plot on the left is a Left-hand Sum with area $L_{50} = 2.3034$, while the right-hand plot is a Midpoint Sum with area $M_{50} = 2.3333$. Notice how much better these approximations are to the actual area of $2.333\ldots$.





Exercise 2: Approximate the area under the graph of the function $f(x) = 1/x$ between $x = 1$ and $x = 4$ by computing L_6, R_6, and M_6. Draw pictures of each sum and determine whether each sum is an overestimate or underestimate. The graph of L_6 is shown on the next page.



We close this section by stating an important fact about the area under the graph of a continuous function. If we let the number of rectangles become arbitrarily large (i.e., let $n \to \infty$), then the Left-hand, Right-hand and Midpoint Sums all limit on the same value A. We define this value to be the area under the curve.

Area Theorem: If $f(x)$ is a continuous function on the interval $[a, b]$, then

$$\lim_{n\to\infty} L_n \;=\; \lim_{n\to\infty} R_n \;=\; \lim_{n\to\infty} M_n \;=\; A\,.$$

We define A to be the area under the graph of $f(x)$ between $x = a$ and $x = b$.