MATH 133 Calculus 1 with FUNdamentals

The Second Derivative and Its Implications

Worksheet for Section 4.4

Key Idea: This section focuses on using the second derivative to understand properties of a function such as where it is concave up or concave down. The second derivative can also be used to determine whether a critical point is a local maximum or minimum. Understanding both the first and second derivative of a function enables us to draw a very detailed graph of a function.

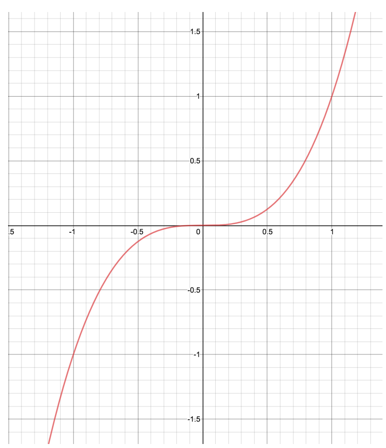
Recall that the second derivative $f''(x)$ indicates whether a function is concave up ($f'' > 0$) or concave down ($f'' < 0$). Concave up means that the first derivative is increasing (slopes increasing), while concave down means that the first derivative is decreasing (slopes decreasing).

$$\boxed{\begin{array}{c} f''(x) > 0 \text{ for } x \in (a,b) \implies f \text{ is concave up on } (a,b) \\ f''(x) < 0 \text{ for } x \in (a,b) \implies f \text{ is concave down on } (a,b) \end{array}}$$

An **inflection point** is a point where the concavity changes. It can be found by solving the equation $f''(x) = 0$ and then checking that the sign of f'' flips to either side of a solution.

Note: It is helpful to draw a second derivative number line and indicate where f'' is positive, negative, or 0.

Example 1: The function $f(x) = x^3$ has an inflection point at $x = 0$. We have $f'(x) = 3x^2$ and $f''(x) = 6x$. Thus, $f''(x) < 0$ for $x < 0$, $f''(x) > 0$ when $x > 0$, and $f''(0) = 0$. The function is concave down for $x < 0$ and concave up for $x > 0$ (see graph below).



Exercise 1: Determine the intervals on which the function $f(x) = xe^{2x}$ is concave up or concave down. Find any points of inflection.

The second derivative can also be used to determine the type of critical point when f'' exists. For example, if $x = c$ is a critical point (so $f'(c) = 0$), and $f''(c) > 0$, then $x = c$ is a local minimum because the function is concave up at $x = c$. Likewise, if $f''(c) < 0$ at a critical point $x = c$, then c is a local max because the graph is concave down.

Second Derivative Test: Suppose that $x = c$ is a critical point of f.

$$\boxed{\begin{array}{ccc} f''(c) > 0 & \Longrightarrow & c \text{ is a local min} \\ f''(c) < 0 & \Longrightarrow & c \text{ is a local max} \\ f''(c) = 0 & \Longrightarrow & \text{test is inconclusive} \end{array}}$$

Note: If c is a critical point and $f''(c) = 0$, then c may either be a local max, a local min, or neither. For instance, the function $f(x) = x^4$ has a local min at the critical point $c = 0$ even though $f''(0) = 0$, while $g(x) = -x^4$ has a local max at $c = 0$ although $g''(0) = 0$.

Exercise 2: Let $g(x) = x^4 - 4x^3$. Find and classify all critical points. Find any inflection points and give the intervals on which the function is concave up or down. Use the first and second derivatives to sketch the graph of g.

Exercise 3: Use the first and second derivatives to sketch the graph of $y = \sin x + \frac{1}{2}x$ over the interval $[0, 2\pi]$. Identify all critical points and inflection points on your graph.