# MATH 133 Calculus 1 with FUNdamentals

## Section 4.5: L'Hôpital's Rule

**Key Idea:** Limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ can be evaluated using a special rule called L'Hôpital's Rule. Simply take the derivative of the numerator and denominator, and then take the limit again. The rule is also useful for finding horizontal asymptotes of a function.

Recall that limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $0 \cdot \infty$ are called **indeterminate forms** because their values can be anything. When you encounter a limit of this type, further work must be done to determine if the limit exists and to find the value of the limit. In 1696, the French mathematician Guillaume François Antoine Marquis de L'Hôpital (pronounced "Lo-pee-tal") published the first textbook on calculus. Included in the book was the following useful technique for computing limits of indeterminate forms:

**L'Hôpital's Rule:** Suppose that $f$ and $g$ are differentiable functions at $x = a$ and that $f(a) = g(a) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right-hand side of the equation exists.

**Notes:**

1. The rule also applies if $a = \infty$.
2. The rule also applies to limits of the form $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$.
3. You may have to apply the rule more than once to determine the limit.
4. The rule was actually discovered by Bernoulli in 1694 but L'Hôpital bought the rights to Bernoulli's discoveries.

**Example 1:** Consider $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$, where, as always, $x$ is assumed to be in radians. Using a calculator or graph, we computed this limit way back in Section 2.6 to be 1. Note that the limit is of the form $\dfrac{0}{0}$, so L'Hôpital's Rule applies. Taking the derivative of the top and bottom, we find

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1,$$

as expected.

**Exercise 1:** Use L'Hôpital's Rule to compute the other important trig limit from Section 2.6:

$$\lim_{x\to 0} \frac{1 - \cos x}{x} =$$

Does this agree with the value we obtained back in Section 2.6?

**Exercise 2:** Use L'Hôpital's Rule to compute the following limits. Be sure to first check that the rule actually applies.

**(a)** $\lim_{x \to 2} \dfrac{x^3 - 8}{x^4 - x^3 - 8}$

**(b)** $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$

**(c)** $\lim_{\theta \to 0} \dfrac{\tan(6\theta)}{\sin(4\theta)}$

### Using L'Hôpital's Rule to find horizontal asymptotes

Recall that if $\lim_{x \to \infty} f(x) = L$, then $f$ has a horizontal asymptote at $y = L$. Since L'Hôpital's Rule applies to limits where $x \to \pm\infty$, we can use it to find any horizontal asymptotes.

**Exercise 3:** Use L'Hôpital's Rule to find any horizontal asymptotes of the following functions.

**(a)** $f(x) = \dfrac{8x^3 - 4x + \pi}{-3x^3 + 7x^2 + 5}$

**(b)** $g(x) = \dfrac{3e^{2x} + 5x}{e^{2x} + 8x}$

**(c)** $h(x) = xe^{-x}$ **Hint:** First write $h(x)$ as a fraction.