MATH 133 Calculus 1 with FUNdamentals

Section 3.6: Trigonometric Functions

In this section we learn how to take the derivative of the standard trig functions. The following key formulas can be proven using the definition of the derivative and the trig identities $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$ and $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$.

Theorem: Assume that x is measured in radians.

$$\boxed{\frac{d}{dx}(\sin x) \;=\; \cos x \qquad \text{and} \qquad \frac{d}{dx}(\cos x) \;=\; -\sin x\,.}$$

To prove the first formula, let $f(x) = \sin x$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \cdot \sin h}{h} \\ &= \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 \;=\; \cos x, \end{aligned}$$

where we have made use of the two important trig limits $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$ (only valid because we are using radians). A similar proof can be used to show that the derivative of $\cos x$ is $-\sin x$.

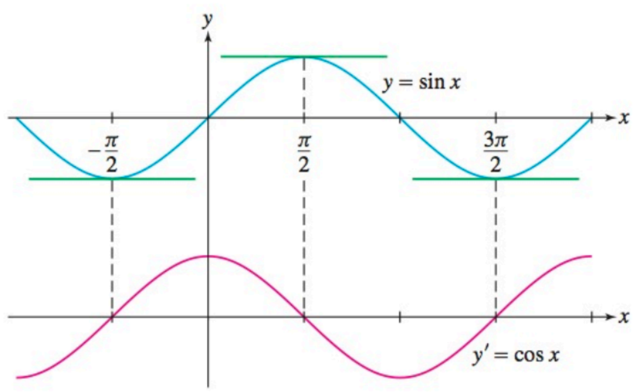

Figure 1: Note that the graph of the derivative of $y = \sin x$ looks exactly like the graph of $\cos x$.

The derivatives of the other four trig functions can now be found using the quotient rule. For instance, we have that $\frac{d}{dx}(\tan x) \;=\; \sec^2 x$ because

$$\frac{d}{dx}(\tan x) \;=\; \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \;=\; \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \;=\; \frac{1}{\cos^2 x} \;=\; \left(\frac{1}{\cos x}\right)^2 \;=\; \sec^2 x.$$

Exercise 1: Use the quotient rule to show that $\dfrac{d}{dx}(\cot x) \;=\; -\csc^2 x$.

Exercise 2: Show that $\dfrac{d}{dx}(\sec x) \;=\; \sec x \cdot \tan x$.

Exercise 3: If $g(x) = x^3 \sin x$, find and simplify $g'(x)$ and $g''(x)$.

Exercise 4: Find the equation of the tangent line to $y = \dfrac{1-3x}{\cos x + x\sin x}$ at $x = 0$.